An algebraic geometry of paths via the iterated-integral signature

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We consider the signature $\sigma(X)$ of a path X as a linear functional on the space of words $T(\mathbb{R}^d)$, e.g.

$$\langle \sigma(X), \mathsf{1232} \rangle = \int_0^T \int_0^s \int_0^u \int_0^t \mathrm{d}X_r^{(1)} \mathrm{d}X_t^{(2)} \mathrm{d}X_u^{(3)} \mathrm{d}X_s^{(2)}.$$

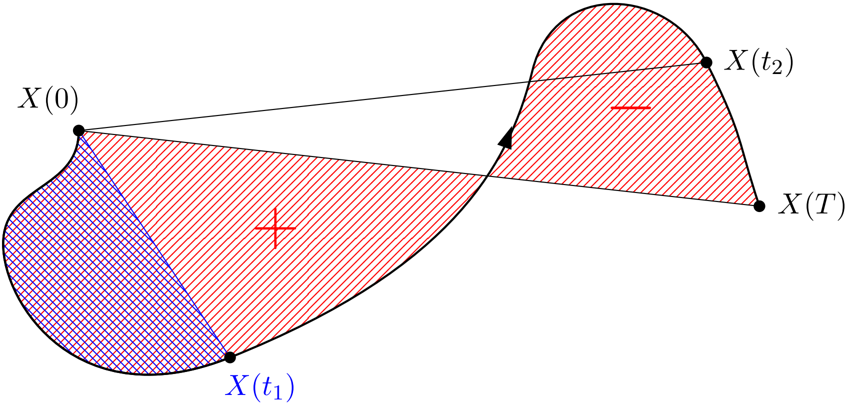
There exists a bilinear operation $\succ : \mathrm{T}^{\geq 1}(\mathbb{R}^d) \times \mathrm{T}^{\geq 1}(\mathbb{R}^d) \to \mathrm{T}^{\geq 1}(\mathbb{R}^d)$ such that

$$\int_0^s \mathbf{X}_t^a \mathrm{d}\mathbf{X}_t^b = \mathbf{X}_s^{a \succ b},$$

where $\mathbf{X}_t^a := \langle \sigma(\mathsf{X}\restriction_{[0,t]}), a \rangle$ [e.g. Gehrig-Kawski 2008], in particular

$$\mathbf{X}_t^a \mathbf{X}_t^b = \mathbf{X}_t^{a \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup b}$$

where $a \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup b := a \succ b + b \succ a$ [Shuffle identity, Ree 1958]



$$\mathsf{SignedArea}(X^1, X^2)_t = \tfrac{1}{2}\Big(\int_0^t X^1_s \mathrm{d}X^2_s - \int_0^t X^2_s \mathrm{d}X^1_s \Big).$$

Picture from Diehl-Lyons-P.-Reizenstein, *Areas of areas generate the shuffle algebra*

The Zariski topology on path space

Let $\mathcal{P}$ be a space of paths such that σ is well-defined on $\mathcal{P}$, e.g. bounded variation paths.
The *antitone Galois connection* $(\mathcal{V}, \mathcal{I})$ is given through $\mathcal{V} : 2^{T(\mathbb{R}^d)} \to 2^{\mathcal{P}}$ being defined as

$$\mathcal{V}(M) = \{Y \in \mathcal{P} | \langle \sigma(Y), x \rangle = 0 \, \forall \, x \in M\}$$

and $\mathcal{I} : 2^{\mathcal{P}} \to 2^{T(\mathbb{R}^d)}$ defined as

$$\mathcal{I}(S) = \{x \in T(\mathbb{R}^d) | \langle \sigma(Y), x \rangle = 0 \, \forall \, Y \in S\}.$$

The closure $\mathrm{cl}_{\mathcal{P}}$ is given by $\mathcal{V} \circ \mathcal{I}$.
The image of $\mathrm{cl}_{\mathcal{P}}$ is then defined to be the collection of closed sets of the *Zariski topology* on $\mathcal{P}$. We also refer to the Zariski closed sets as *varieties*.

How it started

1. Améndola, Friz, Sturmfels: Varieties of signature tensors
2. chats with Francesco Galuppi, Bernd Sturmfels and others
3. Colmenarejo, Preiß: Signatures of paths transformed by polynomial maps
   Our main statement is the relation for a polynomial map p with $p(0) = 0$ that
   $$\langle \sigma(p(Z)), x \rangle = \langle \sigma(Z), M_p(x) \rangle$$
   where M_p can be defined by half-shuffles.
   Important remark: If p 'defines' a classical real affine variety V_p, then the paths living in V_p are basically those Y such that $M_p^* \sigma(Y) = e$.
4. got me thinking

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4. got me thinking (how it's going)

Path algebraic geometry is different from classical algebraic geometry

If a variety V in the image of $\mathrm{cl}_{\mathcal{P}}$ is non-empty and finite dimensional, then $\mathcal{I}(V)$ is infinitely generated as a shuffle ideal.
If V is infinite dimensional, then $\mathcal{I}(V)$ can still be finitely generated or infinitely generated.

Also, the half-shuffle matters. If a variety V is such that it contains all left subpaths, then $\mathcal{I}(V)$ is even a half-shuffle ideal.

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Conjecture: If V is a variety and $\mathcal{I}(V)$ is a half-shuffle ideal, then V contains all left subpaths.

TO DOs

- projective and complex formulations
- topology and tangent bundles on finite and infinite dimensional varieties
- rough paths on classical real affine varieties
- shuffle ideals versus Lie ideals
- algebraic classification of varieties: half-shuffle, antipode, toric etc
- singularities
- semi-algebraic sets, 'integrate' Améndola-Friz-Sturmfels-Galuppi et al
- signatures of classical algebraic curves
- generalized algebraic curves
- applications: invariants, etc

Thank you.

Check out my website: `rosapreiss.net`