

MASTER'S THESIS

From Hopf algebras to rough paths and regularity structures

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Zusammenfassung in deutscher Sprache

Im Mittelpunkt dieser Arbeit stehen die von Lyons entwickelten Rough Paths (wörtlich 'raue Pfade'), die einen algebraischen und analytischen Rahmen für Stieltjes Integrale, also Integrale eines Pfades gegen einen anderen, bieten, in Fällen, in denen die betrachteten Pfade so rau sind, dass das klassische Riemann-Stieltjes Integral nicht mehr konvergiert. Bevor wir Rough Paths formal definieren können, beginnen wir mit einer Einführung in einige algebraische Grundbegriffe. Dazu gehören zunächst Algebren und Koalgebren, zwei Begriffe, die in einer gewissen Dualitätsbeziehung zueinander stehen. Als Kombination der beiden Begriffe erhalten wir Bialgebren, und als Spezialfall dieser wiederum Hopf Algebren, die für diese Arbeit von zentraler Bedeutung sind. Nachdem wir uns mit wichtigen Eigenschaften dieser Objekte vertraut gemacht haben, widmen wir uns einer ausführlichen Diskussion verschiedener Beispiele von Hopf Algebren. Dazu gehört unter anderem die Hopf Algebra der Polynome, bei der das Produkt durch die übliche Multiplikation von Polynomen gegeben ist und sich das Koproduct durch eine vertraute Binomialformel ausdrücken lässt. Anschließend benutzen wir das Beispiel der Tensor Hopf Algebren zur Definition von Weakly Geometric Rough Paths (wörtlich 'schwach geometrische raue Pfade'), die solchen Formen von Stieltjes Integralen entsprechen, welche die übliche Regel der partiellen Integration erfüllen. Für Fälle wie die Itô-Integration, bei denen diese Regel nicht mehr anwendbar ist, betrachten wir die von Gubinelli eingeführten Branched Rough Paths (wörtlich 'verzweigte raue Pfade') und verwenden dazu das Beispiel der Hopf Algebren auf Räumen von als Bäume und Wälder bezeichneten Strukturen. Schließlich gehen wir noch auf einige Grundkonzepte der von Hairer entwickelten Theorie der Regularitätsstrukturen ein, die uns einen weiteren Zugang zu Weakly Geometric und Branched Rough Paths bietet. Dabei betrachten wir zuerst eine von Hairer beschriebene allgemeine Methode, wie sich aus bestimmten Hopf Algebren eine Regularitätsstruktur ableiten lässt, um dann auf Basis einer formalen Picard Iteration eine Regularitätsstruktur zu entwickeln, welche besser zur Betrachtung sogenannter Rough Differential Equations (wörtlich 'raue Differentialgleichungen') geeignet ist.

Abstract

Lyon's rough paths give an algebraic and analytic framework for Stieltjes integrals in a regime of low regularity where the usual Riemann-Stieltjes integral does not converge. Before we may rigorously define rough paths, we start with the introduction of some basic algebraic terminology. Among them are algebras and coalgebras, two notions which are in some sense dual to each other. As a combination of these notions we obtain bialgebras, and as a special case of them then Hopf algebras, which play a central role in this thesis. After further algebraic preliminaries, we give the examples of Hopf algebras we are interested in. Among them is the example of the polynomial Hopf algebra, whose product is nothing but the usual multiplication of polynomials and whose coproduct can be expressed very simply with the help of a binomial coefficient. We then use the dual pair of tensor Hopf algebras to introduce weakly geometric rough paths, which correspond to notions of Stieltjes integrals satisfying the usual integration by parts rule. For cases like Itô-integration where we need to give up integration by parts, we look at Gubinelli's branched rough paths based on the dual pair of Hopf algebras on trees and forests. Finally, we give some basic concepts of Hairer's theory of regularity structures and use them for a different approach to branched and weakly geometric rough paths. While we first look at a general method described by Hairer to derive a regularity structure from certain Hopf algebras, we then develop a regularity structure based on a formal Picard iteration which is more suitable for dealing with rough differential equations.

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Chapter 1

Introduction

Consider a family $(X^i)_{i=1}^d$ of paths $X_i \in \mathcal{C}^\gamma([0, T], \mathbb{R})$, the space of Hölder continuous functions of exponent $\gamma > 1/2$. By the integration theory of Young, introduced in the paper [You36], the Riemann-Stieltjes integrals (Equation (4.1) [FH14])

$$\int_s^t X_r^i dX_r^j := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} X_u^i (X_v^j - X_u^j),$$

where the limit is taken over partitions \mathcal{P} of $[s, t]$, are well defined (cf. Section 3.3.2 [LQ02]). We may even compute *iterated integrals* of the type (Equation (1.3) [HK14])

$$\hat{\mathbf{X}}_{st}(e_{i_1 \dots i_n}) := \int_s^t \cdots \int_s^{r_1} dX_{r_1}^{i_1} \cdots dX_{r_n}^{i_n},$$

where $n \in \mathbb{N}$, $i_1, \dots, i_n \in \{1, \dots, d\}$. The set of all these *words* is denoted by \mathcal{W}_d . Together with later research, most notably by Chen (see e.g. [Che71]), it turns out that these iterated integrals fulfill the following algebraic and analytic conditions (Based on Definition 1.2. [HK14]):

1. $\hat{\mathbf{X}}_{st}(\mathbf{1}) = 1$ and $\hat{\mathbf{X}}_{st}(w_1 \sqcup w_2) = \hat{\mathbf{X}}_{st}(w_1) \hat{\mathbf{X}}_{st}(w_2)$ for all words $w_1, w_2 \in \mathcal{W}_d$,
2. $\hat{\mathbf{X}}_{tt} = \varepsilon$ and $\hat{\mathbf{X}}_{st}(w) = (\hat{\mathbf{X}}_{su} \otimes \hat{\mathbf{X}}_{ut}) \Delta_{\otimes} w = \sum_{(w)}^{\otimes} \hat{\mathbf{X}}_{su}(w^1) \hat{\mathbf{X}}_{ut}(w^2)$ for all words $w \in \mathcal{W}_d$,
3. $\sup_{s \neq t} \frac{|\hat{\mathbf{X}}_{st}(w)|}{|t-s|^{\gamma|w|}} < \infty$ for all words $w \in \mathcal{W}_d$,

where \sqcup is the *shuffle product* and Δ_{\otimes} is the *deconcatenation coproduct* on the real vector space $\langle \mathcal{W}_d \rangle$. The object $(\langle \mathcal{W}_d \rangle, \sqcup, \Delta_{\otimes})$ forms a *bialgebra*. More precisely, it is a special kind of a bialgebra called a *Hopf algebra*.

The crucial point, though, is that we assumed a Hölder exponent $\gamma > 1/2$. Once we have paths of lower Hölder regularity, all of the above breaks down. Our iterated integrals aren't defined any more. This is where *rough paths* come into play. Now that we have $\gamma \in (0, 1/2]$, the idea is to take the above three conditions as a definition, and to say that if a map $\tilde{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{W}_d \rangle^*$ obeys all of them, then we call it a *weakly geometric rough path*.

This is however not the last generalization we want to make. Condition 1. from above encodes nothing but the integration by parts rule applied to iterated integrals. Since we also want to cover theories of integration like Itô-integration which do not satisfy integration by parts, we later give another collection of conditions based on a different Hopf algebra. This leads to the

concept of *branched rough paths*, which was introduced by Gubinelli in the paper [Gub10]. Rough paths as such however were originally developed by Lyons in the papers [Lyo94], [Lyo95] and [Lyo98].

Before we can rigorously introduce rough paths, there are a lot of algebraic concepts to understand. In Chapter 2, we do so in mainly following the notes [Man06], but also repeatedly refer to the book [Abe80]. The standard reference for the following topics is the book [Swe69]. We start by defining *algebras* and *coalgebras*, two notions which are in some sense dual to each other. With certain *compatibility requirements*, they can be combined to the notion of a bialgebra. For some bialgebras, there is a certain algebra and coalgebra antimorphism called the *antipode*, which is characterized by a property involving the so-called *convolution product* algebra given by the product and coproduct. Bialgebras with an antipode are called Hopf algebras. We introduce gradings which make it possible for us to even better understand the structure of the examples of Hopf algebras we are interested in even better. Finally, we formally describe the duality relation already mentioned, but also learn about duality between Hopf algebras as a whole, which was presented in almost the same way in [Che71].

Chapter 3 then presents these examples of Hopf algebras. First, we look at the polynomial Hopf algebra with the usual product of polynomials and a coproduct which is easy to express through a multi dimensional binomial coefficient. A dual Hopf algebra can be described as the algebra of differential operators with composition product, but it turns out to be isomorphic to the polynomial one. Referring to the presentations of the book [Reu93], the paper [HK14] and again [Man06], we then look at the *tensor algebra* whose coproduct is nothing but the dual of the shuffle product mentioned above.

Afterwards, we take a short look at the technicalities of formal series and truncations, and furthermore state some well-known basics about Lie algebras. Also, we introduce the exponential map, which maps a Lie algebra to its Lie group, and its inverse, the logarithm. The exponential map then plays an important role for us by relating *primitive* and *group-like* elements in the case of a connected graded Hopf algebras, as explained in [HK14] and for the special case of the tensor algebra in [Reu93].

As our last example, we look at the Connes-Kreimer Hopf algebra of trees and forests, mainly based on [HK14] and the original paper [CK98]. After introducing all the objects fully recursively, we give representation formulas which formalize the notion of admissible cuts already introduced in [CK98] without relying on interpreting trees as graphs. These formulas make it easy to introduce the dual Hopf algebra whose product is the *Grossman-Larson product*, the dual of the Connes-Kreimer coproduct.

In Chapter 4, we finally get to learn more about weakly geometric and branched rough paths. In both cases, we look at the equivalence of four different kinds of definitions. For that equivalence, the extension theorems of Lyons, which we took from the book [LCL07], and of Gubinelli, found in [Gub10], play a crucial role. Citing the result from [HK14], we will see that weakly geometric rough paths are indeed included in the concept of branched rough paths.

The final Chapter 5 then gives an introduction to some concepts from Hairer's theory of regularity structures which was originally presented in its full scope in the paper [Hai14]. Exactly as described in [Hai14], we look at how a regularity structure can be derived from a connected graded Hopf algebra, and we relate rough paths to *models* for the regularity structure obtained from the respective Hopf algebra. The last section then generalizes a different approach to regularity structures for rough paths which [FH14] shortly looked at for the simplest non-trivial case. It is motivated by a formal Picard iteration, which is a standard method of solving partial differential equations, which we here use for the problem of a rough differential equation.

Chapter 2

Basic definitions and results

For some vector space V over some field K , we denote by V^* its *algebraic dual space*, i.e. the vector space of all linear maps $V \rightarrow K$. If V is equipped with a topology, we write V' for its *topological dual space*, i.e. the vector space of all continuous linear maps $V \rightarrow K$.

Definition 2.1. (Section I.1. [Man06]) Let V_1, V_2 be two vector spaces over the same field K . A *tensor product* $V_1 \otimes V_2$ is a K vector space for which there is a bilinear map $\iota : V_1 \times V_2 \rightarrow V_1 \otimes V_2$ with the property that for every K vector space W and every bilinear map $\rho : V_1 \times V_2 \rightarrow W$, there is a unique linear map $m_\rho : V_1 \otimes V_2 \rightarrow W$ such that

$$\rho = m_\rho \circ \iota. \tag{2.1}$$

We write $v_1 \otimes v_2 := \iota(v_1, v_2)$ for all $v_1 \in V_1, v_2 \in V_2$. With that notation, Equation (2.1) can be formulated as

$$\rho(v_1, v_2) = m_\rho(v_1 \otimes v_2) \quad \forall v_1 \in V_1, v_2 \in V_2.$$

It turns out that for every pair of vector spaces V_1, V_2 over the same field, there exists a tensor product $V_1 \otimes V_2$, and that the tensor product is unique up to linear isomorphisms (Proposition I.1.1. [Man06]). For the proof of existence, one needs the axiom of choice. See the proof of Proposition I.1.1. [Man06] for both existence and uniqueness.

What is very important for us is that for every element $x \in V_1 \otimes V_2$, there is $n \in \mathbb{N}$ and $v_1^1, \dots, v_1^n \in V_1, v_2^1, \dots, v_2^n \in V_2$ such that

$$x = \sum_{i=1}^n \iota(v_1^i, v_2^i) = \sum_{i=1}^n v_1^i \otimes v_2^i.$$

For this fact, also see the proof of Proposition I.1.1. [Man06]. Note that such a representation is of course not unique. Also note that in the literature, one may sometimes find notions of completed tensor products, especially when the vector spaces are endowed with topologies, and for these objects, this fact does not hold true.

Due to the uniqueness of the tensor product up to isomorphisms, we may identify $(V_1 \otimes V_2) \otimes V_3$ with $V_1 \otimes (V_2 \otimes V_3)$, and therefore simply write $V_1 \otimes V_2 \otimes V_3$ (Section I.1. [Man06]).

2.1 Algebras and coalgebras

Definition 2.2. (Section I.2.1. [Man06]) A vector space A together with an associative bilinear map $\diamond : A \times A \rightarrow A$ is called an *algebra*. A subspace $S \subseteq A$ is called *subalgebra* if $S \diamond S \subseteq S$

and *two-sided ideal* if $S \diamond A + A \diamond S \subseteq S$ holds. (A, \diamond) is *unital* if there is a *unit element* $\mathbf{1} \in A$ such that $\mathbf{1} \diamond a = a \diamond \mathbf{1} = a$ for all $a \in A$. The function $u : K \rightarrow A$, $t \mapsto t\mathbf{1}$, where K is the field under A , is called *unit map*.

If an algebra (A, \diamond) is given, we denote by $m_\bullet : A \otimes A \rightarrow A$ the linear map generated by $m_\bullet(a_1 \otimes a_2) := a_1 \diamond a_2$. Using this notation, associativity reads

$$m_\bullet(\text{id} \otimes m_\bullet) = m_\bullet(m_\bullet \otimes \text{id}) \quad (2.2)$$

and the unit map is characterized by

$$m_\bullet(\text{id} \otimes u) = m_\bullet(u \otimes \text{id}) = \text{id}, \quad (2.3)$$

where we use the identification $A = A \otimes K = K \otimes A$ implying $\text{id} \otimes u, u \otimes \text{id} \in L(A, A \otimes A)$.

Definition 2.3. (Chapter 1 Section 2.1 [Abe80]¹) An *algebra homomorphism* between two algebras $(A_1, \diamond), (A_2, \blacklozenge)$ is a linear map $\Lambda : A_1 \rightarrow A_2$ such that $\Lambda m_\bullet = m_\bullet(\Lambda \otimes \Lambda)$.

Definition 2.4. (Section I.3.1. [Man06]) Let C be a vector space together with a linear map $\Delta : C \rightarrow C \otimes C$. If Δ is *coassociative*, i.e.

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta, \quad (2.4)$$

we call (C, Δ) a *coalgebra* and Δ a *coproduct*. A subspace $S \subseteq C$ is called *two-sided coideal* if $\Delta S \subseteq S \otimes C + C \otimes S$ and *subcoalgebra* if $\Delta S \subseteq S \otimes S$. The coalgebra (C, Δ) is *counital* if there is $\varepsilon \in C^*$ such that

$$(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}, \quad (2.5)$$

a so-called *counit*. (C, Δ) is *cocommutative* if $\hat{\tau}\Delta = \Delta$, where $\hat{\tau} : C \otimes C \rightarrow C \otimes C$ is the *flip* linearly generated by $\hat{\tau}(a \otimes b) := b \otimes a$.

Remark 2.5. While every ideal of an algebra is a subalgebra but generally not every subalgebra is an ideal, every subcoalgebra is a coideal but generally not every coideal is a subcoalgebra.

We sometimes write

$$\sum_{(c)} c_1 \otimes c_2 := \Delta c$$

and for arbitrary linear $L : C \otimes C \rightarrow V$ then $\sum_{(c)} L(c_1 \otimes c_2) := L\Delta c$. This concept is called *Sweedler's notation* (Section I.3.1. [Man06]). In Sweedler's notation, coassociativity reads (section I.3.1. [Man06])

$$\sum_{(c)} \sum_{(c_1)} c_{1:1} \otimes c_{1:2} \otimes c_2 = \sum_{(c)} \sum_{(c_2)} c_1 \otimes c_{2:1} \otimes c_{2:2}. \quad (2.6)$$

Definition 2.6. (Chapter 2 Section 1.1 [Abe80]²) A *coalgebra homomorphism* between two coalgebras $(C_1, \Delta_1), (C_2, \Delta_2)$ is a linear map $\Lambda : C_1 \rightarrow C_2$ such that $(\Lambda \otimes \Lambda)\Delta_1 = \Delta_2\Lambda$.

¹In [Abe80], the unit property is included in the definition of an algebra, and thus also the definition of an algebra homomorphism, which is simply called an algebra morphism there, contains the condition that the unit of one algebra is mapped to the unit of the other algebra.

²As with the definition of an algebra, the definition of a coalgebra in [Abe80] already includes the counit property and thus also the definition of a coalgebra morphism there includes an additional condition concerning the counits.

If we have an algebra (A, \diamond) and a coalgebra (C, Δ) , the *convolution product* $*$ (Section I.4. [Man06]) on the space of linear maps $L(C, A)$ is defined by

$$S * T := m_\circ(S \otimes T)\Delta. \quad (2.7)$$

$(L(C, A), *)$ forms an algebra, as

$$\begin{aligned} S * (T * V) &= m_\circ(S \otimes (m_\circ(T \otimes V)\Delta))\Delta \\ &= m_\circ(\text{id} \otimes m_\circ)(S \otimes T \otimes V)(\text{id} \otimes \Delta)\Delta = m_\circ(m_\circ \otimes \text{id})(S \otimes T \otimes V)(\Delta \otimes \text{id})\Delta \\ &= m_\circ((m_\circ(S \otimes T)\Delta) \otimes V)\Delta = (S * T) * V, \end{aligned}$$

where we used associativity of \diamond and coassociativity of Δ (based on Section I.4. [Man06]). If (A, \diamond) is unitary with unit map u and (C, Δ) is counitary with counit ε , then $(L(C, A), *)$ is unitary with unit element $u\varepsilon$ (based on Section I.5. [Man06]) as

$$T * u\varepsilon = m_\circ(T \otimes u\varepsilon)\Delta = m_\circ(\text{id} \otimes u)T(\text{id} \otimes \varepsilon)\Delta = T \quad (2.8)$$

and analogously $u\varepsilon * T = T$.

Definition 2.7. (Section I.3.2. [Man06]) Let (C, Δ) be a coalgebra over the field K . Then, a *right comodule* is a K vector space \hat{C} together with a map $\hat{\Delta} : \hat{C} \rightarrow C \otimes \hat{C}$ such that

$$(\Delta \otimes \text{id})\hat{\Delta} = (\text{id} \otimes \hat{\Delta})\hat{\Delta}.$$

2.2 Bialgebras and Hopf algebras

For an algebra (A, \diamond) we introduce another algebra $(A \otimes A, \diamond)$ where the product is bilinearly generated by (Section I.2.2. [Man06])

$$(a_1 \otimes a_2) \diamond (a_3 \otimes a_4) := (a_1 \diamond a_3) \otimes (a_2 \diamond a_4). \quad (2.9)$$

The corresponding linear map is given by (Section I.2.2. [Man06])

$$m_\diamond = (m_\circ \otimes m_\circ) \underbrace{(\text{id} \otimes \hat{\tau} \otimes \text{id})}_{=:\hat{\tau}_{1324}}.$$

Definition 2.8. (Section I.5. [Man06], Chapter 2 Section 1.1 [Abe80]) Let (B, \diamond) be an algebra over the field K with unit element $\mathbf{1} = u(1)$ and (B, Δ) a coalgebra with counit ε . If furthermore the *compatibility requirements*

1. $\Delta m_\circ = (m_\circ \otimes m_\circ)(\text{id} \otimes \hat{\tau} \otimes \text{id})(\Delta \otimes \Delta)$ (Δ is an algebra homomorphism from (B, \diamond) to $(B \otimes B, \diamond)$ and \diamond is a coalgebra homomorphism from $(B \otimes B, \hat{\tau}_{1324}(\Delta \otimes \Delta))$ to (B, Δ)),
2. $\Delta u = u \otimes u$ (u is a coalgebra homomorphism from (K, id_K) to (B, Δ)),
3. $\varepsilon m_\circ = \varepsilon \otimes \varepsilon$ (ε is an algebra homomorphism from (B, \diamond) to (K, \cdot)),

hold, we call (B, \diamond, Δ) a *bialgebra*. For bialgebras $(B_1, \diamond_1, \Delta_1)$ with unit map u_1 and counit ε_1 , $(B_2, \diamond_2, \Delta_2)$ with unit map u_2 and counit ε_2 , we call $\Lambda : B_1 \rightarrow B_2$ a *bialgebra homomorphism* if Λ is both an algebra homomorphism and a coalgebra homomorphism and fulfills the additional properties $\Lambda u_1 = u_2$ and $\varepsilon_2 \Lambda = \varepsilon_1$.

Remark 2.9. Applied to elements of B , the compatibility requirements read

1. $\Delta(b_1 \diamond b_2) = \Delta b_1 \diamond \Delta b_2 \quad \forall b_1, b_2 \in B$,
2. $\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$,
3. $\varepsilon(b_1 \diamond b_2) = \varepsilon(b_1)\varepsilon(b_2) \quad \forall b_1, b_2 \in B$.

Remark 2.10. In a bialgebra, we always have $\varepsilon(\mathbf{1}) = 1$, as the counit property (2.5) implies

$$\mathbf{1} = (\text{id} \otimes \varepsilon)\Delta \mathbf{1} = (\text{id} \otimes \varepsilon)(\mathbf{1} \otimes \mathbf{1}) = \varepsilon(\mathbf{1})\mathbf{1}.$$

Thus, $\varepsilon u = \text{id}_K$ and therefore $\varepsilon = \varepsilon u \varepsilon$.

We introduce the following simple lemma to avoid having to show this separately in each of the examples of Hopf algebras we later want to look at.

Lemma 2.11. *Let (A, \diamond) be an algebra and $\Delta : A \rightarrow A \otimes A$ an algebra homomorphism. Assume that*

$$(\Delta \otimes \text{id})\Delta v = (\text{id} \otimes \Delta)\Delta v \tag{2.10}$$

holds for all $v \in M \subseteq A$. Then, we have (2.10) for all v in the subalgebra generated by M . If furthermore $\varepsilon \in A^$ is an algebra homomorphism and*

$$(\varepsilon \otimes \text{id})\Delta v = (\text{id} \otimes \varepsilon)\Delta v = v \tag{2.11}$$

holds for all $v \in M$, we even have (2.11) for all v in the subalgebra generated by M .

Proof. Let S denote the subalgebra generated by M . As Δ is an algebra homomorphism, we have for all v, w in S that

$$\Delta(v \diamond w) = \sum_{(v)} \sum_{(w)} (v_1 \diamond w_1) \otimes (v_2 \diamond w_2)$$

and therefore, if v and w satisfy (2.10), we get, using the notation from (2.6),

$$\begin{aligned} (\Delta \otimes \text{id})\Delta(v \diamond w) &= \sum_{(v)} \sum_{(v_1)} \sum_{(w)} \sum_{(w_1)} (v_{1:1} \diamond w_{1:1}) \otimes (v_{1:2} \diamond w_{1:2}) \otimes (v_2 \diamond w_2) \\ &= \sum_{(v)} \sum_{(v_2)} \sum_{(w)} \sum_{(w_1)} (v_1 \diamond w_{1:1}) \otimes (v_{2:1} \diamond w_{1:2}) \otimes (v_{2:2} \diamond w_2) \\ &= \sum_{(v)} \sum_{(v_2)} \sum_{(w)} \sum_{(w_2)} (v_1 \diamond w_1) \otimes (v_{2:1} \diamond w_{2:1}) \otimes (v_{2:2} \diamond w_{2:2}) \\ &= (\text{id} \otimes \Delta)\Delta(v \diamond w). \end{aligned}$$

As the equation (2.10) is furthermore linear in v , which follows from Δ being an algebra homomorphism, coassociativity indeed inductively extends from M to the whole of S .

If also ε is an algebra morphism, the counit property given for some $v, w \in S$ implies

$$\begin{aligned} (\varepsilon \otimes \text{id})\Delta(v \diamond w) &= \sum_{(v)} \sum_{(w)} \varepsilon(v_1 \diamond w_1)(v_2 \diamond w_2) = \sum_{(v)} \sum_{(w)} (\varepsilon(v_1)v_2) \diamond (\varepsilon(w_1)w_2) \\ &= \left(\sum_{(v)} \varepsilon(v_1)v_2 \right) \diamond \left(\sum_{(w)} \varepsilon(w_1)w_2 \right) = v \diamond w. \end{aligned}$$

Analogously, we get $(\text{id} \otimes \varepsilon)\Delta(v \diamond w) = v \diamond w$. Hence, also the counit property inductively extends from M to S . □

For a bialgebra (B, \diamond, Δ) with unit element $\mathbf{1}$, let

$$\sum_{(h)} h' \otimes h'' := \tilde{\Delta} h := \Delta h - \mathbf{1} \otimes h - h \otimes \mathbf{1}$$

denote the *reduced coproduct* of h (Proof of Proposition II.1.1. [Man06], Section 2.2 [HK14]).

Lemma 2.12. (Part of Proposition II.1.1. [Man06]) $(B, \tilde{\Delta})$ is a coalgebra.

Proof. (Part of the proof of Proposition II.1.1. [Man06]) As Δ maps B linearly to $B \otimes B$, the same is true for $\tilde{\Delta}$. Thus, according to Definition 2.4, the only thing left to show is coassociativity. By Definition 2.8 of a bialgebra we have $\Delta u = u \otimes u$, implying

$$\tilde{\Delta} u = -u \otimes u. \quad (2.12)$$

Therefore, we compute

$$\begin{aligned} (\tilde{\Delta} \otimes \text{id})\tilde{\Delta} &= (\tilde{\Delta} \otimes \text{id})(\Delta - u \otimes \text{id} - \text{id} \otimes u) \stackrel{(2.12)}{=} (\tilde{\Delta} \otimes \text{id})\Delta + u \otimes u \otimes \text{id} - \tilde{\Delta} \otimes u \\ &= (\Delta \otimes \text{id})\Delta + u \otimes u \otimes \text{id} + u \otimes \text{id} \otimes u + \text{id} \otimes u \otimes u - \Delta \otimes u - (\text{id} \otimes u \otimes \text{id})\Delta - u \otimes \Delta \\ &\stackrel{(2.4)}{=} (\text{id} \otimes \Delta)\Delta + u \otimes u \otimes \text{id} + u \otimes \text{id} \otimes u + \text{id} \otimes u \otimes u - \Delta \otimes u - (\text{id} \otimes u \otimes \text{id})\Delta - u \otimes \Delta \\ &= (\text{id} \otimes \tilde{\Delta})\Delta + \text{id} \otimes u \otimes u - u \otimes \tilde{\Delta} \stackrel{(2.12)}{=} (\text{id} \otimes \tilde{\Delta})\tilde{\Delta}. \end{aligned}$$

□

Having shown that $\tilde{\Delta}$ is a coproduct, we may introduce the convolution product algebra $(L(B, B), \tilde{*})$ by

$$S \tilde{*} T := m_*(S \otimes T)\tilde{\Delta}. \quad (2.13)$$

Definition 2.13. (Section I.5. [Man06]) A bialgebra (H, \diamond, Δ) with unit map u and counit ε together with an *antipode*, i.e. a linear operator $\mathcal{S} : H \rightarrow H$ fulfilling the *antipode property*

$$m_*(\mathcal{S} \otimes \text{id})\Delta = m_*(\text{id} \otimes \mathcal{S})\Delta = \underbrace{u\varepsilon}_{=\varepsilon(\cdot)\mathbf{1}}, \quad (2.14)$$

is called a *Hopf algebra*.

Although [Abe80] gives the definition of a Hopf algebra homomorphism (morphism in the language used there), the following simple fact we came across is not mentioned, but it is for example stated as Exercise 9. of Section III.8 [Kas95]. The idea for the proof came from a similar argumentation in the proof of Proposition I.7.1. [Man06].

Theorem 2.14. Let $(H_1, \diamond, \Delta_1, \mathcal{S}_1)$ and $(H_2, \blacklozenge, \Delta_2, \mathcal{S}_2)$ be Hopf algebras and $\Lambda : H_1 \rightarrow H_2$ be a bialgebra homomorphism. Then,

$$\Lambda \mathcal{S}_1 = \mathcal{S}_2 \Lambda.$$

Therefore, we call Λ a Hopf algebra homomorphism (Chapter 2 Section 1.2 [Abe80]).

Proof. Consider the algebra $(L(H_1, H_2), *)$ defined by $S * T := m_*(S \otimes T)\Delta_1$ with unit $u_2\varepsilon_1$ (see (2.7) and (2.8)), where u_i is the unit map and ε_i is the counit of H_i . We have

$$\begin{aligned} \Lambda \mathcal{S}_1 * \Lambda &= m_*(\Lambda \mathcal{S}_1 \otimes \Lambda)\Delta_1 = m_*(\Lambda \otimes \Lambda)(\mathcal{S}_1 \otimes \text{id})\Delta_1 = \Lambda m_*(\mathcal{S}_1 \otimes \text{id})\Delta_1 = \Lambda u_1\varepsilon_1 = u_2\varepsilon_1 \\ &= u_2\varepsilon_2 \Lambda = m_*(\text{id} \otimes \mathcal{S}_2)\Delta_2 \Lambda = m_*(\text{id} \otimes \mathcal{S}_2)(\Lambda \otimes \Lambda)\Delta_1 = m_*(\Lambda \otimes \mathcal{S}_2 \Lambda)\Delta_1 = \Lambda * \mathcal{S}_2 \Lambda \end{aligned}$$

and therefore

$$\Lambda \mathcal{S}_1 = \Lambda \mathcal{S}_1 * u_2\varepsilon_1 = \Lambda \mathcal{S}_1 * (\Lambda * \mathcal{S}_2 \Lambda) = (\Lambda \mathcal{S}_1 * \Lambda) * \mathcal{S}_2 \Lambda = u_2\varepsilon_1 * \mathcal{S}_2 \Lambda = \mathcal{S}_2 \Lambda. \quad (2.15)$$

□

Corollary 2.15. (Section 2.1 [HK14]) *If there is an antipode for a given bialgebra, then it is unique.*

Proof. If for a bialgebra (B_1, \diamond, Δ) , there are antipodes \mathcal{S}_1 and \mathcal{S}_2 , then both $(B_1, \diamond, \Delta, \mathcal{S}_1)$ and $(B_2, \diamond, \Delta, \mathcal{S}_2)$ are Hopf algebras, thus we have $\mathcal{S}_1 = \mathcal{S}_2$ via Theorem 2.14 using the trivial bialgebra homomorphism id . \square

Theorem 2.16. (Proposition 4.0.1. [Swe69], Theorem III.3.4. [Kas95], Proposition I.7.1. [Man06]) *Let $(H, \diamond, \Delta, \mathcal{S})$ be a Hopf algebra with unit element $\mathbf{1}$, unit map u and counit ε . The antipode \mathcal{S} is*

(i) *an algebra antimorphism, i.e.*

$$\mathcal{S}m_* = m_*(\mathcal{S} \otimes \mathcal{S})\hat{\tau}, \quad (2.16)$$

where $\hat{\tau}$ denotes the flip, resp.

$$\mathcal{S}(h_1 \diamond h_2) = \mathcal{S}h_2 \diamond \mathcal{S}h_1 \quad \forall h_1, h_2 \in H.$$

Furthermore, $\mathcal{S}u = u$, resp. $\mathcal{S}\mathbf{1} = \mathbf{1}$.

(ii) *a coalgebra antimorphism, i.e.*

$$\Delta\mathcal{S} = \hat{\tau}(\mathcal{S} \otimes \mathcal{S})\Delta. \quad (2.17)$$

Furthermore, $\varepsilon\mathcal{S} = \varepsilon$.

Proof. (Based on the proofs of Proposition 4.0.1. [Swe69], of Theorem III.3.4. [Kas95] and of Proposition I.7.1. and Lemma I.7.2. [Man06]) First of all, using associativity (2.2), we have

$$\begin{aligned} m_*(m_* \otimes m_*) &= m_*(m_* \otimes \text{id})(\text{id} \otimes \text{id} \otimes m_*) = m_*(\text{id} \otimes m_*)(\text{id} \otimes \text{id} \otimes m_*) \\ &= m_*(\text{id} \otimes m_*)(\text{id} \otimes m_* \otimes \text{id}). \end{aligned} \quad (2.18)$$

Likewise, using coassociativity (2.4),

$$\begin{aligned} (\Delta \otimes \Delta)\Delta &= (\text{id} \otimes \text{id} \otimes \Delta)(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \text{id} \otimes \Delta)(\text{id} \otimes \Delta)\Delta \\ &= (\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)\Delta. \end{aligned} \quad (2.19)$$

Put $\underline{\Delta} := (\text{id} \otimes \hat{\tau} \otimes \text{id})(\Delta \otimes \Delta)$. Furthermore, for any permutation $(i_1 \dots i_n)$, let $\hat{\tau}_{i_1 \dots i_n} : H^{\otimes n} \rightarrow H^{\otimes n}$ be the linear map generated by (Section I.1. [Man06])

$$\hat{\tau}_{i_1 \dots i_n}(h_{i_1} \otimes \dots \otimes h_{i_n}) := h_1 \otimes \dots \otimes h_n.$$

(i) Consider the algebra $(L(H \otimes H, H), *)$ defined by $S * T := m_*(S \otimes T)\underline{\Delta}$ with unit $u(\varepsilon \otimes \varepsilon)$. Using the antipode property (2.14) and the unit property (2.3), we get

$$\begin{aligned} m_* * m_*(\mathcal{S} \otimes \mathcal{S})\hat{\tau} &= m_*(m_* \otimes m_*(\mathcal{S} \otimes \mathcal{S})\hat{\tau})\underline{\Delta} = m_*(m_* \otimes m_*(\mathcal{S} \otimes \mathcal{S})\hat{\tau})\hat{\tau}_{1423}(\Delta \otimes \Delta) \\ &\stackrel{(2.18)}{=} m_*(\text{id} \otimes m_*)(\text{id} \otimes m_* \otimes \text{id})(\text{id} \otimes \text{id} \otimes \mathcal{S} \otimes \mathcal{S})\hat{\tau}_{1342}(\Delta \otimes \Delta) \\ &= m_*(\text{id} \otimes m_*)\hat{\tau}_{132}(\text{id} \otimes \text{id} \otimes m_*)(\text{id} \otimes \mathcal{S} \otimes \text{id} \otimes \mathcal{S})(\Delta \otimes \Delta) \\ &\stackrel{(2.14)}{=} m_*(\text{id} \otimes m_*)\hat{\tau}_{132}(\text{id} \otimes \mathcal{S} \otimes u\varepsilon)(\Delta \otimes \text{id}) \\ &\stackrel{(2.3)}{=} m_*(\text{id} \otimes \mathcal{S})(\Delta \otimes \varepsilon) \stackrel{(2.14)}{=} u\varepsilon \otimes \varepsilon = u(\varepsilon \otimes \varepsilon). \end{aligned}$$

On the other hand, using the antipode property as well as the first and third compatibility requirement in Definition 2.8, we have

$$\mathcal{S}m_* * m_* = m_*(\mathcal{S}m_* \otimes m_*)\underline{\Delta} = m_*(\mathcal{S} \otimes \text{id})(m_* \otimes m_*)\underline{\Delta} = m_*(\mathcal{S} \otimes \text{id})\Delta m_* = u\varepsilon m_* = u(\varepsilon \otimes \varepsilon).$$

Then, equation (2.16) follows completely analogous to the calculation in (2.15).

By Remark 2.10, the antipode property, the second compatibility requirement and the unit property we also have

$$\mathbf{1} = u\varepsilon\mathbf{1} = m_*(\mathcal{S} \otimes \text{id})\Delta\mathbf{1} = \mathcal{S}\mathbf{1} \diamond \mathbf{1} = \mathcal{S}\mathbf{1}.$$

- (ii) Consider the algebra $(L(H, H \otimes H), *)$ defined by $S * T := m_*(S \otimes T)\Delta$ with unit $(u \otimes u)\varepsilon$. Using the antipode property (2.14) and the counit property (2.3), we get

$$\begin{aligned} \Delta * \hat{\tau}(\mathcal{S} \otimes \mathcal{S})\Delta &= m_*(\Delta \otimes \hat{\tau}(\mathcal{S} \otimes \mathcal{S})\Delta)\Delta = (m_* \otimes m_*)\hat{\tau}_{1324}(\Delta \otimes \hat{\tau}(\mathcal{S} \otimes \mathcal{S})\Delta)\Delta \\ &\stackrel{(2.19)}{=} (m_* \otimes m_*)\hat{\tau}_{1342}(\text{id} \otimes \text{id} \otimes \mathcal{S} \otimes \mathcal{S})(\text{id} \otimes \Delta \otimes \text{id})(\text{id} \otimes \Delta)\Delta \\ &= (m_* \otimes m_*)(\text{id} \otimes \mathcal{S} \otimes \text{id} \otimes \mathcal{S})(\text{id} \otimes \text{id} \otimes \Delta)\hat{\tau}_{132}(\text{id} \otimes \Delta)\Delta \\ &\stackrel{(2.14)}{=} (m_* \otimes \text{id})(\text{id} \otimes \mathcal{S} \otimes u\varepsilon)\hat{\tau}_{132}(\text{id} \otimes \Delta)\Delta \\ &\stackrel{(2.5)}{=} (m_* \otimes u)(\text{id} \otimes \mathcal{S})\Delta \stackrel{(2.14)}{=} u\varepsilon \otimes u = (u \otimes u)\varepsilon. \end{aligned}$$

On the other hand, using the antipode property as well as the first and second compatibility requirement in Definition 2.8, we have

$$\Delta\mathcal{S} * \Delta = m_*(\Delta\mathcal{S} \otimes \Delta)\Delta = m_*(\Delta \otimes \Delta)(\mathcal{S} \otimes \text{id})\Delta = \Delta m_*(\mathcal{S} \otimes \text{id})\Delta = \Delta u\varepsilon = (u \otimes u)\varepsilon.$$

Then, equation (2.17) follows completely analogous to the calculation in (2.15).

By Remark 2.10, the antipode property, the third compatibility requirement and the counit property we also have

$$\varepsilon = \varepsilon u\varepsilon = \varepsilon m_*(\mathcal{S} \otimes \text{id})\Delta = (\varepsilon \otimes \varepsilon)(\mathcal{S} \otimes \text{id})\Delta = \varepsilon\mathcal{S}(\text{id} \otimes \varepsilon)\Delta = \varepsilon\mathcal{S}.$$

□

2.3 Gradings

Definition 2.17. (Chapter 1 Section 2.2 and Chapter 2 Section 4.1 [Abe80], Section II.1. [Man06]) Let $G_i, i \in \mathbb{N}_0$ be vector spaces over the same field and put $G := \bigoplus_{i=0}^{\infty} G_i$. Then,

- (i) an algebra (G, \diamond) is *graded* if

$$G_n \diamond G_m \subseteq G_{n+m} \quad \forall n, m \in \mathbb{N}_0,$$

- (ii) a coalgebra (G, Δ) is *graded* if

$$\Delta G_n \subseteq \bigoplus_{m \leq n} G_m \otimes G_{n-m} \quad \forall n \in \mathbb{N}_0,$$

- (iii) a bialgebra (G, \diamond, Δ) is *graded* if both (G, \diamond) and (G, Δ) are graded,

- (iv) a Hopf algebra $(G, \diamond, \Delta, \mathcal{S})$ is *graded* if (G, \diamond, Δ) is graded and

$$\mathcal{S}G_n \subseteq G_n \quad \forall n \in \mathbb{N}_0.$$

The family $(G_i)_{i \in \mathbb{N}_0}$ is called *grading* of G . The algebra, coalgebra, bialgebra or Hopf algebra is *connected graded* if additionally $\dim G_0 = 1$.

If $(G_i)_{i \in \mathbb{N}_0}$ is a grading, the family $(G^i)_{i \in \mathbb{N}_0}$, $G^i := \bigoplus_{j=0}^i G_j$ is a so-called *filtration* (Chapter 1 Section 2.2 and Chapter 2 Section 4.1 [Abe80], Section II.2. [Man06]).

Theorem 2.18. (Proposition II.1.1. [Man06]) *In a connected graded bialgebra (B, \diamond, Δ) graded by $(B_i)_i$ we have $\ker \varepsilon = \bigoplus_{i=1}^{\infty} B_i$ and*

$$\tilde{\Delta} B_n \subseteq \bigoplus_{0 < m < n} B_m \otimes B_{n-m} \quad \forall n \in \mathbb{N}. \quad (2.20)$$

Proof. (Proof of Proposition II.1.1. [Man06]) Let K be the field under B . Let $n \in \mathbb{N}$ and $x \in B_n$ be arbitrary. From Definition 2.17 we know that $\Delta x \in \bigoplus_{m \leq n} B_m \otimes B_{n-m}$. As furthermore $B_0 = \text{span}\{\mathbf{1}\}$ due to connectedness, there are $y, z \in B_n$ and $w \in \bigoplus_{0 < m < n} B_m \otimes B_{n-m}$ such that

$$\Delta x = \mathbf{1} \otimes y + z \otimes \mathbf{1} + w.$$

By the counit property (2.5) and $\varepsilon(\mathbf{1}) = 1$ (see Remark 2.10), we have

$$x = (\text{id} \otimes \varepsilon)\Delta x = \varepsilon(y)\mathbf{1} + z + (\text{id} \otimes \varepsilon)w.$$

Due to linear independence of $z - x \in B_n$, $\varepsilon(y)\mathbf{1} \in B_0$ and $(\text{id} \otimes \varepsilon)w \in \bigoplus_{0 < m < n} B_m$ this implies $x = z$, $\varepsilon(y) = 0$ and $(\text{id} \otimes \varepsilon)w = 0$. Again applying the counit property, we get, using what we just found out,

$$x = (\varepsilon \otimes \text{id})\Delta x = \varepsilon(x)\mathbf{1} + y + (\varepsilon \otimes \text{id})w.$$

As before, linear independence of $y - x \in B_n$, $\varepsilon(x)\mathbf{1} \in B_0$ and $(\varepsilon \otimes \text{id})w \in \bigoplus_{0 < m < n} B_m$ implies $x = y$, $\varepsilon(x) = 0$ and $(\varepsilon \otimes \text{id})w = 0$. Hence, we conclude

$$\Delta x = \mathbf{1} \otimes x + x \otimes \mathbf{1} + w,$$

implying $\tilde{\Delta} x = w \in \bigoplus_{0 < m < n} B_m \otimes B_{n-m}$, as well as $\varepsilon(x) = 0$. As $n \in \mathbb{N}$ and $x \in B_n$ where arbitrary, this implies (2.20) and, together with $\varepsilon(\mathbf{1}) = 1$, also $\ker \varepsilon = \bigoplus_{i=1}^{\infty} B_i$. \square

Remark 2.19. Due to Theorem 2.18 we especially have

$$\Delta b_1 = \mathbf{1} \otimes b_1 + b_1 \otimes \mathbf{1} \quad \forall b_1 \in B_1.$$

The following theorem will be very important for some of our examples of Hopf algebras later. We chose a different approach for the proof than the one given in [Man06]. For a function $f : M \rightarrow N$ and $M' \subseteq M$, let $f|_{M'}$ denote the restriction of f to M' .

Theorem 2.20. (Corollary II.3.2. [Man06]) *Any connected graded bialgebra (H, \diamond, Δ) over the field K is a connected graded Hopf algebra with the antipode given by the recursion*

$$\mathcal{S}h = -h - m_*(\text{id} \otimes \mathcal{S})\tilde{\Delta}h = -h - \sum_{(h)} h' \diamond \mathcal{S}h'' \quad \forall h \in \ker \varepsilon, \quad \mathcal{S}k\mathbf{1} = k\mathbf{1} \quad \forall k \in K, \quad (2.21)$$

or equivalently

$$\mathcal{S}h = -h - m_*(\mathcal{S} \otimes \text{id})\tilde{\Delta}h = -h - \sum_{(h)} \mathcal{S}h' \diamond h'' \quad \forall h \in \ker \varepsilon, \quad \mathcal{S}k\mathbf{1} = k\mathbf{1} \quad \forall k \in K. \quad (2.22)$$

Proof. Let $(H_i)_i$ be the grading of H , $(H^i)_i$ the filtration and define the family $(\tilde{H}^i)_{i \in \mathbb{N}}$ by $\tilde{H}^i := \bigoplus_{j=1}^i H_j$.

1. *The individual recursions are well-defined.* We only look at (2.21), the other case is completely analogous. We proceed by induction. First of all, $\mathcal{S}|_{H_0}$ is obviously a well-defined linear operator as H is connected. Assume $\mathcal{S}|_{H^n}$ is a well-defined linear operator. As (2.20) implies $\tilde{\Delta}H^{n+1} \subseteq \tilde{H}^n \otimes \tilde{H}^n$, the operator

$$\mathcal{S}|_{H^{n+1}} := -\text{id}|_{H^{n+1}} - m_*(\text{id} \otimes \mathcal{S}|_{H^n})\tilde{\Delta}|_{H^{n+1}}$$

is well-defined in accordance with (2.21) and linear due to the fact that it is a sum of compositions of linear operators.

2. *Both recursions yield the same results.* Let \mathcal{S}_1 be the linear operator obtained by (2.21) and \mathcal{S}_2 the one obtained by (2.22). Again, we proceed by induction. As H is connected, $H_0 = \text{span}\{\mathbf{1}\}$ and hence $\mathcal{S}_1|_{H_0} = \mathcal{S}_2|_{H_0}$ by definition. Assume

$$\mathcal{S}_1|_{H^n} = \mathcal{S}_2|_{H^n} \tag{2.23}$$

for some $n \in \mathbb{N}_0$. Then,

$$\begin{aligned} (\text{id} \tilde{*} \mathcal{S}_1)|_{\tilde{H}^{n+1}} &\stackrel{(\diamond)}{=} (\text{id} \tilde{*} \mathcal{S}_2)|_{\tilde{H}^{n+1}} = -(\text{id} \tilde{*} \text{id})|_{\tilde{H}^{n+1}} - (\text{id} \tilde{*} \mathcal{S}_2 \tilde{*} \text{id})|_{\tilde{H}^{n+1}} \\ &\stackrel{(\diamond)}{=} -(\text{id} \tilde{*} \text{id})|_{\tilde{H}^{n+1}} - (\text{id} \tilde{*} \mathcal{S}_1 \tilde{*} \text{id})|_{\tilde{H}^{n+1}} = (\mathcal{S}_1 \tilde{*} \text{id})|_{\tilde{H}^{n+1}} \stackrel{(\diamond)}{=} (\mathcal{S}_2 \tilde{*} \text{id})|_{\tilde{H}^{n+1}} \end{aligned}$$

where we used (2.23) and the definition of $\tilde{*}$ (2.13) combined with $\tilde{\Delta}H^{n+1} \subseteq \tilde{H}^n \otimes \tilde{H}^n$ for the equalities labeled (\diamond) . We conclude $\mathcal{S}_1|_{H^{n+1}} = \mathcal{S}_2|_{H^{n+1}}$.

3. *Both recursions yield the antipode.* Now, let $\mathcal{S} := \mathcal{S}_1 = \mathcal{S}_2$ be the operator given by each of the recursions. For $h \in \ker \varepsilon$, we have

$$m_*(\mathcal{S} \otimes \text{id})\Delta h = m_*(\mathcal{S} \otimes \text{id})\tilde{\Delta}h + m_*(\mathcal{S} \otimes \text{id})(\mathbf{1} \otimes h + h \otimes \mathbf{1}) = m_*(\mathcal{S} \otimes \text{id})\tilde{\Delta}h + h + \mathcal{S}h \stackrel{(2.22)}{=} 0$$

and analogously $m_*(\text{id} \otimes \mathcal{S})\Delta h = 0$ by (2.22). Also, $u\varepsilon(h) = 0$. Hence, the antipode property (2.14) is fulfilled on $\ker \varepsilon$. Finally, due to the fact that $\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ in a bialgebra, we get

$$m_*(\mathcal{S} \otimes \text{id})\Delta \mathbf{1} = m_*(\mathcal{S} \mathbf{1} \otimes \mathbf{1}) = \mathbf{1} = m_*(\text{id} \otimes \mathcal{S})\Delta \mathbf{1},$$

just as $u\varepsilon(\mathbf{1}) = \mathbf{1}$. By linearity, the antipode property is fulfilled on the whole of H .

Finally, we show by yet another induction that \mathcal{S} preserves the grading. First of all, we obviously have $\mathcal{S}H_0 = H_0$. Assuming $\mathcal{S}H_m \subseteq H_m$ for all $m < n$, we get $\mathcal{S}H_n \subseteq H_n$ by (2.21) and

$$m_*(\text{id} \otimes \mathcal{S})\tilde{\Delta}H_n \stackrel{(2.20)}{\subseteq} m_*(\text{id} \otimes \mathcal{S}) \bigoplus_{0 < m < n} H_m \otimes H_{n-m} \subseteq m_* \bigoplus_{0 < m < n} H_m \otimes H_{n-m} \subseteq H_n$$

since (H, \diamond, Δ) was assumed to be a graded bialgebra. □

2.4 Pairs of dual vector spaces

Definition 2.21. (Chapter II Section 3. [RR64], Chapter 2 Section 2.1 [Abe80]) Let V, \tilde{V} be two vector spaces over the same field K and $\langle \cdot, \cdot \rangle : V \times \tilde{V} \rightarrow K$ bilinear such that

1. $\langle v, \tilde{v} \rangle = 0 \forall \tilde{v} \in \tilde{V} \implies v = 0$,
2. $\langle v, \tilde{v} \rangle = 0 \forall v \in V \implies \tilde{v} = 0$.

Then, we call (V, \tilde{V}) a *pair of dual vector spaces* and $\langle \cdot, \cdot \rangle$ its *duality pairing*.

Lemma 2.22. (Chapter II Section 5. [RR64]) Let (V, \tilde{V}) be a pair of dual vector spaces and V finite dimensional. Then, $\dim V = \dim \tilde{V}$ and for each basis $(b_i)_i$ of V there is a dual basis $(\tilde{b}_i)_i$ of \tilde{V} , i.e. $\langle b_i, \tilde{b}_j \rangle = \delta_{i,j}$.

Proof. (Based on Chapter II Section 5. [RR64]) For each $\tilde{v} \in \tilde{V}$ let $\iota(\tilde{v}) := \langle \cdot, \tilde{v} \rangle \in V'$. Obviously, ι is linear. If $\tilde{v} \neq 0$, then there is $v \in V$ such that $\langle v, \tilde{v} \rangle \neq 0$ as $\langle \cdot, \cdot \rangle$ is a duality pairing. Hence $\iota(\tilde{v}) \neq 0$, i.e. ι is injective. Therefore $\dim \tilde{V} \leq \dim V' = \dim V$. Thus \tilde{V} is also finite dimensional and by the same argument as before we get $\dim V \leq \dim \tilde{V}' = \dim \tilde{V}$. As $\iota : \tilde{V} \rightarrow V'$ is injective and $\dim \tilde{V} = \dim V'$ we have that ι is bijective. Let now $(b_i)_i$ be a basis of V and define $\kappa_i \in V'$ by $\kappa_i(b_j) := \delta_{i,j}$. Then $(\iota^{-1}(\kappa_i))_i$ is a dual basis of $(b_i)_i$. \square

Theorem 2.23. For $i \in \{1, 2\}$, let $\langle \cdot, \cdot \rangle_i$ be a duality pairing of a K -vector space V_i with another K -vector space \tilde{V}_i , where K is a field of characteristic zero. Then, $(V_1 \otimes V_2, \tilde{V}_1 \otimes \tilde{V}_2)$ is a pair of dual vector spaces with the induced duality pairing

$$\left\langle \sum_{j=1}^m v_1^j \otimes v_2^j, \sum_{l=1}^n \tilde{v}_1^l \otimes \tilde{v}_2^l \right\rangle := \sum_{j=1}^m \sum_{l=1}^n \langle v_1^j, \tilde{v}_1^l \rangle_1 \langle v_2^j, \tilde{v}_2^l \rangle_2. \quad (2.24)$$

Proof.

1. The map $\langle \cdot, \cdot \rangle$ is well-defined and bilinear. Let $m_i : V_i \otimes \tilde{V}_i \rightarrow K$ be the linear maps corresponding to $\langle \cdot, \cdot \rangle_i$. Put $m := (m_1 \otimes m_2)(\text{id} \otimes \hat{\tau} \otimes \text{id})$. Then, $\langle \cdot, \cdot \rangle := m(\cdot \otimes \cdot)$ yields the expansion (2.24) and is obviously bilinear.
2. The map $\langle \cdot, \cdot \rangle$ is a duality pairing. Let $0 \neq u = \sum_{j=1}^m u_1^j \otimes u_2^j \in V_1 \otimes V_2$ be arbitrary. Without loss of generality, we may assume that

$$\dim \text{span}(u_1^j)_{1 \leq j \leq m} = \dim \text{span}(u_2^j)_{1 \leq j \leq m} = m,$$

because otherwise the number of summands may be reduced. Put $W_i := \text{span}(u_1^j)_{1 \leq j \leq m}$,

$$W_i^\perp := \{\tilde{v} \in \tilde{V}_i \mid \langle w, \tilde{v} \rangle_i = 0 \forall w \in W_i\}$$

and $F_i := \tilde{V}_i / W_i^\perp$. For $\tilde{v} \in \tilde{V}_i$ let $[\tilde{v}] \in F_i$ denote its equivalence class. Hence, for $\tilde{v}, \tilde{v}' \in \tilde{V}_i$,

$$[\tilde{v}] = [\tilde{v}'] \iff \langle w, \tilde{v} \rangle_i = \langle w, \tilde{v}' \rangle_i \forall w \in W_i.$$

This means that $\langle w, [\tilde{v}] \rangle'_i := \langle w, \tilde{v} \rangle_i$ is well-defined. $\langle w, [\tilde{v}] \rangle'_i = 0$ for all $w \in W_i$ implies by definition $[\tilde{v}] = [0] = 0$. Also, $\langle w, [\tilde{v}] \rangle'_i = 0$ for all $[\tilde{v}] \in F_i$ means by definition $\langle w, \tilde{v} \rangle_i = 0$ for all $\tilde{v} \in \tilde{V}_i$ and therefore $w = 0$ because (V_i, \tilde{V}_i) is a pair of dual vector spaces. Hence, (W_i, F_i) is a pair of dual vector spaces and since W_i is finite dimensional, we may use Lemma 2.22 to get that there is a basis $([\tilde{u}_i^j])_{1 \leq j \leq n}$ of F_i such that

$$\delta_{j,l} = \langle u_i^j, [\tilde{u}_i^l] \rangle'_i = \langle u_i^j, \tilde{u}_i^l \rangle_i.$$

Put $\tilde{u} := \sum_{j=1}^m u_1^j \otimes u_2^j$. Then,

$$\langle u, \tilde{u} \rangle = \sum_{j,l=1}^m \langle u_1^j, \tilde{u}_1^l \rangle_1 \langle u_2^j, \tilde{u}_2^l \rangle_2 = m \neq 0.$$

As $0 \neq u \in V_1 \otimes V_2$ was arbitrary, we just showed implication 1 of Definition 2.21. Showing implication 2 of the definition is then completely analogous. \square

Definition 2.24. Let (V_i, \tilde{V}_i) , $i \in \{1, 2\}$ be pairs of dual vector spaces with duality pairings $\varpi_i = \langle \cdot, \cdot \rangle_i$. We then say that a linear map $S : V_1 \rightarrow V_2$ is the *dual operator* of a linear map $\tilde{S} : \tilde{V}_2 \rightarrow \tilde{V}_1$ if

$$\langle Sv, \tilde{v} \rangle_2 = \langle v, \tilde{S}\tilde{v} \rangle_1$$

for all $v \in V_1$ and $\tilde{v} \in \tilde{V}_2$. We write $S \overset{\varpi_2}{\uparrow\downarrow} \tilde{S}$, or shortly $S \uparrow\downarrow \tilde{S}$ if the involved duality pairings are clear from the context³.

Lemma 2.25. Let (V_i, \tilde{V}_i) and (W_j, \tilde{W}_j) be pairs of dual vector spaces over the same field K .

- (i) If both $S_1 : V_1 \rightarrow V_2$ and $S_2 : V_1 \rightarrow V_2$ are dual operators of $\tilde{S} : \tilde{V}_2 \rightarrow \tilde{V}_1$, then $S_1 = S_2$.
- (ii) If K is of characteristic zero, $A : V_1 \rightarrow V_2$ is the dual operator of $\tilde{A} : \tilde{V}_2 \rightarrow \tilde{V}_1$ and $B : W_1 \rightarrow W_2$ is the dual operator of $\tilde{B} : \tilde{W}_2 \rightarrow \tilde{W}_1$, then $A \otimes B$ is the dual operator of $\tilde{A} \otimes \tilde{B}$ under the induced duality pairing.
- (iii) If $F : V_1 \rightarrow V_2$ is the dual operator of $\tilde{F} : \tilde{V}_2 \rightarrow \tilde{V}_1$ and $G : V_2 \rightarrow V_3$ is the dual operator of $\tilde{G} : \tilde{V}_3 \rightarrow \tilde{V}_2$, then GF is the dual operator of $\tilde{F}\tilde{G}$.

Proof. Let $\langle \cdot, \cdot \rangle_{\nu_i}$ be the duality pairing of (V_i, \tilde{V}_i) and $\langle \cdot, \cdot \rangle_{\omega_j}$ the duality pairing of (W_j, \tilde{W}_j) .

- (i) $\langle S_1 v, \tilde{v} \rangle_{\nu_2} = \langle v, \tilde{S}\tilde{v} \rangle_{\nu_1} = \langle S_2 v, \tilde{v} \rangle_{\nu_2}$ for all $v \in V_1$, $\tilde{v} \in \tilde{V}_2$ implies $\langle (S_1 - S_2)v, \tilde{v} \rangle_{\nu_2} = 0$ for all $v \in V_1$, $\tilde{v} \in \tilde{V}_2$, hence $(S_1 - S_2)v = 0$ for all $v \in V_1$ by the Definition 2.21 of dual pairs.
- (ii) Let $\langle \cdot, \cdot \rangle_{\nu_i \mu_i}$ be the induced duality pairing of $(V_i \otimes W_i, \tilde{V}_i \otimes \tilde{W}_i)$. Under the given assumption,

$$\begin{aligned} \langle (A \otimes B)(v_1 \otimes w_1), \tilde{v}_2 \otimes \tilde{w}_2 \rangle_{\nu_2 \omega_2} &= \langle Av_1, \tilde{v}_2 \rangle_{\nu_2} \langle Bw_1, \tilde{w}_2 \rangle_{\omega_2} = \langle v_1, \tilde{A}\tilde{v}_2 \rangle_{\nu_1} \langle w_1, \tilde{B}\tilde{w}_2 \rangle_{\omega_1} \\ &= \langle v_1 \otimes w_1, (\tilde{A} \otimes \tilde{B})(\tilde{v}_2 \otimes \tilde{w}_2) \rangle_{\nu_1 \omega_1} \end{aligned}$$

for all $v_1 \in V_1$, $w_1 \in W_1$, $\tilde{v}_2 \in \tilde{V}_2$, $\tilde{w}_2 \in \tilde{W}_2$. The implication follows via bilinearity.

- (iii) Under the given assumption,

$$\langle GFv_1, \tilde{v}_3 \rangle_3 = \langle Fv_1, \tilde{G}\tilde{v}_3 \rangle_2 = \langle v_1, \tilde{F}\tilde{G}\tilde{v}_3 \rangle_1$$

for all $v_1 \in V_1$ and $\tilde{v}_3 \in \tilde{V}_3$. \square

³This notation is nonstandard and just used for the purpose of this thesis.

Theorem 2.26. (Equations (3.4.1) to (3.4.5) and Section 3.5 [Che71], also based on Chapter 2 Sections 2 and 3 [Abe80], Proposition I.3.1. [Man06]) *Let (V, \tilde{V}) be a pair of dual vector spaces over the field K of characteristic zero.*

1. *If $\Delta_\circ : \tilde{V} \rightarrow \tilde{V} \otimes \tilde{V}$ is the dual operator of $m_\circ : V \otimes V \rightarrow V$, then $(\tilde{V}, \Delta_\circ)$ is a coalgebra iff (V, \diamond) is an algebra. In that case, we say that $(\tilde{V}, \Delta_\circ)$ is the dual coalgebra of (V, \diamond) and (V, \diamond) is the dual algebra of $(\tilde{V}, \Delta_\circ)$.*
2. *If (V, \diamond) is the dual algebra of $(\tilde{V}, \Delta_\circ)$ and $u : K \rightarrow V$ is the dual operator of $\varepsilon : \tilde{V} \rightarrow K$, then u is a unit map iff ε is a counit.*
3. *If (V, \diamond) is the dual unital algebra of $(\tilde{V}, \Delta_\circ)$ and (V, Δ_\circ) is the dual counital coalgebra of (\tilde{V}, \diamond) , then $(V, \diamond, \Delta_\circ)$ is a bialgebra iff $(\tilde{V}, \diamond, \Delta_\circ)$ is a bialgebra. In that case, we say that $(V, \diamond, \Delta_\circ)$ is the dual bialgebra of $(\tilde{V}, \diamond, \Delta_\circ)$ and vice versa.*
4. *If $(V, \diamond, \Delta_\circ)$ is the dual bialgebra of $(\tilde{V}, \diamond, \Delta_\circ)$ and $\mathcal{S}_\circ^* : V \rightarrow V$ is the dual map of $\mathcal{S}_\circ^* : \tilde{V} \rightarrow \tilde{V}$, then $(\tilde{V}, \diamond, \Delta_\circ, \mathcal{S}_\circ^*)$ is a Hopf algebra iff $(\tilde{V}, \diamond, \Delta_\circ, \mathcal{S}_\circ^*)$ is a Hopf algebra. In that case, we say that $(\tilde{V}, \diamond, \Delta_\circ, \mathcal{S}_\circ^*)$ is the dual Hopf algebra of $(\tilde{V}, \diamond, \Delta_\circ, \mathcal{S}_\circ^*)$ and vice versa.*

Proof.

1. Due to Lemma 2.25 (ii) and (iii),

$$(\text{id} \otimes \Delta_\circ) \Delta_\circ \upharpoonright m_\circ(\text{id} \otimes m_\circ) \quad \text{and} \quad (\Delta_\circ \otimes \text{id}) \Delta_\circ \upharpoonright m_\circ(m_\circ \otimes \text{id}).$$

Hence, by Lemma 2.25 (i), coassociativity of Δ_\circ (cf. (2.4)) and associativity of \diamond (cf. (2.2)) are equivalent.

2. Due to Lemma 2.25 (ii) and (iii),

$$(\text{id} \otimes \varepsilon) \Delta_\circ \upharpoonright m_\circ(\text{id} \otimes u) \quad \text{and} \quad (\varepsilon \otimes \text{id}) \Delta_\circ \upharpoonright m_\circ(u \otimes \text{id}).$$

Hence, by Lemma 2.25 (i), the counit property of ε (cf. (2.5)) and the unit property of u (cf. (2.3)) are equivalent.

3. Due to Lemma 2.25 (ii) and (iii),

$$\Delta_\circ m_\circ \upharpoonright \Delta_\circ m_\circ \quad \text{and} \quad (m_\circ \otimes m_\circ)(\text{id} \otimes \hat{\tau} \otimes \text{id})(\Delta_\circ \otimes \Delta_\circ) \upharpoonright (m_\circ \otimes m_\circ)(\text{id} \otimes \hat{\tau} \otimes \text{id})(\Delta_\circ \otimes \Delta_\circ)$$

as well as

$$\Delta_\circ u \upharpoonright \varepsilon m_\circ \quad \text{and} \quad u \otimes u \upharpoonright \varepsilon \otimes \varepsilon.$$

as well as

$$\varepsilon m_\circ \upharpoonright \Delta_\circ u \quad \text{and} \quad \varepsilon \otimes \varepsilon \upharpoonright u \otimes u.$$

Hence, by Lemma 2.25 (i), the compatibility requirements (cf. definition 2.8) of $\diamond, \Delta_\circ, u_\circ, \varepsilon_\circ$ and those of $\diamond, \Delta_\circ, u_\circ, \varepsilon_\circ$ are equivalent.

4. Due to Lemma 2.25 (ii) and (iii),

$$m_\circ(\text{id} \otimes \mathcal{S}_\circ^*) \Delta_\circ \upharpoonright m_\circ(\text{id} \otimes \mathcal{S}_\circ^*) \Delta_\circ, \quad m_\circ(\mathcal{S}_\circ^* \otimes \text{id}) \Delta_\circ \upharpoonright m_\circ(\mathcal{S}_\circ^* \otimes \text{id}) \Delta_\circ \quad \text{and} \quad u_\circ \varepsilon_\circ \upharpoonright u_\circ \varepsilon_\circ.$$

Hence, by Lemma 2.25 (i), the antipode property (cf. (2.14)) of \mathcal{S}_\circ^* and the antipode property of \mathcal{S}_\circ^* are equivalent.

□

Chapter 3

Connected graded Hopf algebras

3.1 Polynomials and differential operators

Definition 3.1. For the set $\mathbb{N}_0^d = \{n = (n_1, \dots, n_d) | n_i \in \mathbb{N}_0 \forall 1 \leq i \leq d\}$ of d -dimensional *multi indices* we define

- (i) the partial order $n \leq m \iff n_i \leq m_i \forall i$ with the shorthand notation $n < m \iff n \leq m \wedge n \neq m$,
- (ii) the absolute value $|n| := \sum_{i=1}^d n_i$,
- (iii) the sum $+$: $\mathbb{N}_0^d \times \mathbb{N}_0^d \rightarrow \mathbb{N}_0^d$, $n + m := (n_1 + m_1, \dots, n_d + m_d)$,
- (iv) the difference $n - m := (n_1 - m_1, \dots, n_d - m_d)$ for $m \leq n$,
- (v) the factorial $\cdot !$: $\mathbb{N}_0^d \rightarrow \mathbb{N}$, $n! := \prod_{i=1}^d n_i!$,
- (vi) the binomial coefficient $\binom{n}{m} := \frac{n!}{(n-m)!m!}$ for $m \leq n$.

By $(e^i)_{1 \leq i \leq d} \subset \mathbb{N}_0^d$ we denote the family of multi indices given by $e_j^i := \delta_{i,j}$.

3.1.1 Overview

In this section, we want to introduce the following two structures and show that they are Hopf algebras:

$\mathcal{P}_d = \text{span}\{X^n, n \in \mathbb{N}_0^d\} = \mathbb{R}[X_1, \dots, X_d]$	
$\bullet : \mathcal{P}_d \times \mathcal{P}_d \rightarrow \mathcal{P}_d,$	$X^n \bullet X^m = X^{n+m}$
$u : \mathbb{R} \rightarrow \mathcal{P}_d,$	$u(r) = rX^0$
$\Delta_\circ : \mathcal{P}_d \rightarrow \mathcal{P}_d \otimes \mathcal{P}_d,$	$\Delta_\circ X^n = \sum_{0 \leq m \leq n} \binom{n}{m} X^m \otimes X^{n-m}$
$\varepsilon : \mathcal{P}_d \rightarrow \mathbb{R},$	$\varepsilon(X^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$
$\mathcal{S}_\bullet : \mathcal{P}_d \rightarrow \mathcal{P}_d,$	$\mathcal{S}_\bullet X^n = (-1)^n X^n$

$\mathbb{D}_d = \text{span}\{\mathcal{D}^n, n \in \mathbb{N}_0^d\}$	
$\circ : \mathbb{D}_d \times \mathbb{D}_d \rightarrow \mathbb{D}_d,$	$\mathcal{D}^n \circ \mathcal{D}^m = \binom{n+m}{m} \mathcal{D}^{n+m}$
$u : \mathbb{R} \rightarrow \mathbb{D}_d,$	$u(r) = r\mathcal{D}^0$
$\Delta_\bullet : \mathbb{D}_d \rightarrow \mathbb{D}_d \otimes \mathbb{D}_d,$	$\Delta_\bullet \mathcal{D}^n = \sum_{0 \leq m \leq n} \mathcal{D}^m \otimes \mathcal{D}^{n-m}$
$\varepsilon : \mathbb{D}_d \rightarrow \mathbb{R},$	$\varepsilon(\mathcal{D}^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$
$\mathcal{S}_\circ : \mathbb{D}_d \rightarrow \mathbb{D}_d,$	$\mathcal{S}_\circ \mathcal{D}^n = (-1)^n \mathcal{D}^n$

3.1.2 Polynomial algebra

The content of this subsection is standard knowledge and belongs to the folklore of the theory of graded Hopf algebras.

The *polynomial algebra* is defined as the real vector space $\mathcal{P}_d := \text{span}\{X^n, n \in \mathbb{N}_0^d\}$ together with the bilinear product \bullet generated by $X^n X^m := X^n \bullet X^m := X^{n+m}$. Put $\mathbf{1} := X^0$ and $X_i := X^{e^i}$. Our goal is to extend (\mathcal{P}_d, \bullet) to a connected graded Hopf algebra. Intuitively, we choose the grade of some monomial X^n to be equal to its polynomial degree $|n|$. This grading yields the decomposition $\mathcal{P}_d = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_{d,n}$, where $\mathcal{P}_{d,n} := \text{span}\{X^m, m \in \mathbb{N}_0^d : |m| = n\}$. Hence, $\mathcal{P}_{d,n} \bullet \mathcal{P}_{d,m} \subseteq \mathcal{P}_{d,n+m}$ and $\dim \mathcal{P}_{d,0} = 1$, which means that (\mathcal{P}_d, \bullet) is a connected graded algebra (see Definition 2.17). When defining a compatible coproduct Δ_\circ which preserves the grading, we have to have $\Delta_\circ \mathbf{1} := \mathbf{1} \otimes \mathbf{1}$, which is exactly the compatibility requirement for the unit element in a bialgebra (see Remark 2.9), as well as $\Delta_\circ X_i := \mathbf{1} \otimes X_i + X_i \otimes \mathbf{1}$ for all $i \in \{1, \dots, d\}$ due to the grading and remark 2.19. Then, in order to get a coproduct which is an algebra homomorphism as required in a bialgebra (see Definition 2.8), we need to define

$$\Delta_\circ X^n := \prod_{i \in \{1, \dots, d\}} (\Delta_\circ X_i)^{\bullet n_i} = \sum_{0 \leq m \leq n} \binom{n}{m} X^m \otimes X^{n-m} \quad \forall n \in \mathbb{N}_0^d, \quad (3.1)$$

where again $(x \otimes v) \bullet (y \otimes w) := (x \bullet y) \otimes (v \bullet w)$. The coproduct on the whole of \mathcal{P}_d is then generated by linearity.

Remark 3.2. As we just showed there is only one candidate for a coproduct which may extend (\mathcal{P}_d, \bullet) to a bialgebra under the predefined grading $(\mathcal{P}_{d,n})_n$.

Lemma 3.3. $(\mathcal{P}_d, \bullet, \Delta_\circ)$ is a connected graded bialgebra.

Proof. (\mathcal{P}_d, \bullet) is an unital algebra as \bullet is obviously associative and by definition bilinear with unit $\mathbf{1}$. Likewise, Δ_\circ is linear by definition. Its linearity, the bilinearity of \bullet and the first equality in (3.1) then imply that Δ_\circ is an algebra homomorphism. Hence, by Lemma 2.11 coassociativity already follows generally from the special cases

$$\begin{aligned} (\text{id} \otimes \Delta_\circ) \Delta_\circ X_i &= \mathbf{1} \otimes \Delta_\circ X_i + X_i \otimes \Delta_\circ \mathbf{1} = \mathbf{1} \otimes \mathbf{1} \otimes X_i + \mathbf{1} \otimes X_i \otimes \mathbf{1} + X_i \otimes \mathbf{1} \otimes \mathbf{1} \\ &= \Delta_\circ \mathbf{1} \otimes X_i + \Delta_\circ X_i \otimes \mathbf{1} = (\Delta_\circ \otimes \text{id}) \Delta_\circ X_i. \end{aligned}$$

Thus, $(\mathcal{P}_d, \Delta_\circ)$ is a coalgebra. The fact that $\varepsilon \in \mathcal{P}_d^* : \varepsilon(P) = P(0)$ is a counit is obvious from the expansion in (3.1). The validity of the compatibility requirements for the counit and the unit

element, namely $\Delta_\circ \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ and $\varepsilon(P)\varepsilon(Q) = P(0)Q(0) = (PQ)(0) = \varepsilon(PQ)$ for all $P, Q \in \mathcal{P}_d$ (cf. Definition 2.8 and Remark 2.9) then completes the proof of $(\mathcal{P}_d, \bullet, \Delta_\circ)$ being a bialgebra. As $\mathcal{P}_{d,n} \bullet \mathcal{P}_{d,m} \subseteq \mathcal{P}_{d,n+m}$ and

$$\Delta_\circ \mathcal{P}_{d,n} \subseteq \bigoplus_{m+l=n} \mathcal{P}_{d,m} \otimes \mathcal{P}_{d,l}$$

(see (3.1)) as well as $\dim \mathcal{P}_{d,0} = 1$, it is also connected graded according to Definition 2.17. \square

As the graded bialgebra $(\mathcal{P}_d, \bullet, \Delta_\circ)$ is connected, Theorem 2.20 shows that an antipode is given by the recursion

$$\mathcal{S}_\bullet^\circ P = -P - \mathfrak{m}_\bullet(\text{id} \otimes \mathcal{S}_\bullet^\circ) \tilde{\Delta}_\circ P = -P - \sum_{(P)} P' \bullet \mathcal{S}_\bullet^\circ P'' \quad \forall P \in \ker \varepsilon, \quad \mathcal{S}_\bullet^\circ \mathbf{1} = \mathbf{1}. \quad (3.2)$$

Knowing that an antipode exists, we may deduce its form without performing the recursion, but simply from the algebra antimorphism property any antipode fulfills due to Theorem 2.16. Indeed, we just need to compute

$$\begin{aligned} 0 &= u\varepsilon(X_i) = \mathfrak{m}_\bullet(\text{id} \otimes \mathcal{S}_\bullet^\circ) \Delta_\circ X_i = \mathfrak{m}_\bullet(\text{id} \otimes \mathcal{S}_\bullet^\circ)(\mathbf{1} \otimes X_i + X_i \otimes \mathbf{1}) = \mathbf{1} \bullet \mathcal{S}_\bullet^\circ X_i + X_i \bullet \mathcal{S}_\bullet^\circ \mathbf{1} \\ &= \mathcal{S}_\bullet^\circ X_i + X_i, \end{aligned}$$

concluding $\mathcal{S}_\bullet^\circ X_i = -X_i$ for all $i \in \{1, \dots, d\}$, and the fact that $\mathcal{S}_\bullet^\circ$ is a \bullet antimorphism, and thus a homomorphism since \bullet is commutative, already implies that

$$\begin{aligned} \mathcal{S}_\bullet^\circ X^n &= \mathcal{S}_\bullet^\circ (X_1^{n_1} \bullet \dots \bullet X_d^{n_d}) = (\mathcal{S}_\bullet^\circ X_1)^{\bullet n_1} \bullet \dots \bullet (\mathcal{S}_\bullet^\circ X_d)^{\bullet n_d} = (-X_1)^{\bullet n_1} \bullet \dots \bullet (-X_d)^{\bullet n_d} \\ &= (-1)^{|n|} X^n \end{aligned}$$

for all $n \in \mathbb{N}_0^d$. The map $\mathcal{S}_\bullet^\circ$ is then obviously graded, which is also stated by Theorem 2.20. We thus have the following.

Theorem 3.4. *$(\mathcal{P}_d, \bullet, \Delta_\circ, \mathcal{S}_\bullet^\circ)$ is a connected graded Hopf algebra.*

3.1.3 Algebra of differential operators

Now, we consider the space of differential operators $\mathbb{D}_d := \text{span}\{\partial^n, n \in \mathbb{N}_0^d\}$ over the field \mathbb{R} . Our treatment of the corresponding Hopf algebra is based on Section 2.1 [Bro04] and on Example 2.2 [HK14], and also on some standard facts.

The tuple $(\mathcal{P}_d, \mathbb{D}_d)$ is a pair of dual vector spaces via the duality pairing (Remark 4.19 [Hai14])

$$\langle \partial^n, p \rangle := (\partial^n p)(0), \quad p \in \mathcal{P}_d \quad (3.3)$$

The dual basis of $\{X^n, n \in \mathbb{N}_0^d\}$ is then given by $\{\mathcal{D}^n, n \in \mathbb{N}_0^d\}$, where $\mathcal{D}^n := \frac{1}{n!} \partial^n$. Indeed, we have

$$\langle \mathcal{D}^n, X^m \rangle = \frac{1}{n!} (\partial^n X^m)(0) = \delta_{n,m} \quad \forall n, m \in \mathbb{N}_0^d.$$

Analogous to the case of the polynomials, we introduce the intuitive grading leading to the decomposition $\mathbb{D}_d = \bigoplus_{n \in \mathbb{N}} \mathbb{D}_{d,(n)}$ into subspaces $\mathbb{D}_{d,(n)} := \text{span}\{\mathcal{D}^m, m \in \mathbb{N}_0^d : |m| = n\}$ as well as a bilinear product, the composition \circ , via $\partial^n \partial^m := \partial^n \circ \partial^m := \partial^{n+m}$. Obviously, there is an isomorphism $\iota_{\mathcal{P}_d \mathbb{D}_d} : (\mathcal{P}_d, \bullet) \rightarrow (\mathbb{D}_d, \circ)$ such that $\iota_{\mathcal{P}_d \mathbb{D}_d}(X^n) = \partial^n$, i.e. the grading is preserved by it as well. Hence, by Remark 3.2, we also have to choose the coproduct Δ_\bullet and the antipode $\mathcal{S}_\bullet^\circ$

isomorphic to Δ_\circ and $\mathcal{S}_\circ^\bullet$ in order to extend (\mathbb{D}_d, \circ) to a connected graded Hopf algebra, meaning (Section 2.1 [Bro04], Example 2.2 [HK14])

$$\Delta_\bullet \partial^n := \sum_{0 \leq m \leq n} \binom{n}{m} \partial^m \otimes \partial^{n-m}, \quad \mathcal{S}_\circ^\bullet \partial^n := (-1)^{|n|} \partial^n.$$

When using the basis $(\mathcal{D}^n)_{n \in \mathbb{N}_0^d}$, the algebraic rules read

$$\mathcal{D}^n \circ \mathcal{D}^m = \binom{n+m}{m} \mathcal{D}^{n+m}, \quad \Delta_\bullet \mathcal{D}^n = \sum_{0 \leq m \leq n} \mathcal{D}^m \otimes \mathcal{D}^{n-m}, \quad \mathcal{S}_\circ^\bullet \mathcal{D}^n = (-1)^{|n|} \mathcal{D}^n.$$

Theorem 3.5. $(\mathbb{D}_d, \circ, \Delta_\bullet, \mathcal{S}_\circ^\bullet)$ is a connected graded Hopf algebra and dual to $(\mathcal{P}_d, \bullet, \Delta_\circ, \mathcal{S}_\circ^\bullet)$, i.e.

$$\langle D_1 \circ D_2, P \rangle = \langle D_1 \otimes D_2, \Delta_\circ P \rangle \quad \forall D_1, D_2 \in \mathbb{D}_d, P \in \mathcal{P}_d, \quad (3.4)$$

$$\langle \Delta_\bullet D, P \otimes Q \rangle = \langle D, P \bullet Q \rangle \quad \forall D \in \mathbb{D}_d, P, Q \in \mathcal{P}_d, \quad (3.5)$$

$$\langle \mathcal{S}_\circ^\bullet D, P \rangle = \langle D, \mathcal{S}_\circ^\bullet P \rangle \quad \forall D \in \mathbb{D}_d, P \in \mathcal{P}_d. \quad (3.6)$$

Proof. The first claim is guaranteed by Theorem 3.4 and isomorphy by construction. As for the duality identities, it is sufficient to check their validity for the elements of the bases $(X^n)_{n \in \mathbb{N}_0^d}$ and $(\mathcal{D}^n)_{n \in \mathbb{N}_0^d}$ due to the bilinearity of the duality pairing and the products and the linearity of the coproducts and the antipodes. Considering (3.4), we indeed compute

$$\begin{aligned} \langle \mathcal{D}^n \circ \mathcal{D}^m, X^l \rangle &= \binom{n+m}{n} \delta_{n+m, l} = \sum_{0 \leq k \leq l} \binom{l}{k} \delta_{n, k} \delta_{m, l-k} = \sum_{0 \leq k \leq l} \binom{l}{k} \langle \mathcal{D}^n \otimes \mathcal{D}^m, X^k \otimes X^{l-k} \rangle \\ &= \langle \mathcal{D}^n \otimes \mathcal{D}^m, \Delta_\circ X^l \rangle \end{aligned}$$

for all $n, m, l \in \mathbb{N}_0^d$. Likewise,

$$\langle \Delta_\bullet \mathcal{D}^n, X^m \otimes X^l \rangle = \sum_{0 \leq k \leq n} \langle \mathcal{D}^k \otimes \mathcal{D}^{n-k}, X^m \otimes X^l \rangle = \sum_{0 \leq k \leq n} \delta_{k, m} \delta_{n-k, l} = \delta_{n, m+l} = \langle \mathcal{D}^n, X^m \bullet X^l \rangle$$

proves (3.5) and

$$\langle \mathcal{S}_\circ^\bullet \mathcal{D}^n, X^m \rangle = (-1)^{|n|} \delta_{n, m} = (-1)^{|m|} \delta_{n, m} = \langle \mathcal{D}^n, \mathcal{S}_\circ^\bullet X^m \rangle$$

validates (3.6). □

At this point, we would like to move to the algebraic dual space \mathcal{P}_d^* , the space of formal series $\sum_{n \in \mathbb{N}_0^d} a_n \mathcal{D}^n$ of which \mathbb{D}_d is just the subspace of finite series. We can extend the Hopf algebra operations via

$$\begin{aligned} \left(\sum_{n \in \mathbb{N}_0^d} a_n \mathcal{D}^n \right) \circ \left(\sum_{m \in \mathbb{N}_0^d} b_m \mathcal{D}^m \right) &:= \sum_{n, m \in \mathbb{N}_0^d} a_n b_m \mathcal{D}^n \circ \mathcal{D}^m = \sum_{k \in \mathbb{N}_0^d} \sum_{0 \leq l \leq k} \binom{k}{l} a_l b_{k-l} \mathcal{D}^k, \\ \Delta_{\mathcal{P}^*} \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{D}^n &:= \sum_{n \in \mathbb{N}_0^d} a_n \Delta_\bullet \mathcal{D}^n = \sum_{m, k \in \mathbb{N}_0^d} a_{m+k} \mathcal{D}^m \otimes \mathcal{D}^k, \\ \mathcal{S}_{\mathcal{P}'} \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{D}^n &:= \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{S}_\circ^\bullet \mathcal{D}^n = \sum_{n \in \mathbb{N}_0^d} (-1)^{|n|} a_n \mathcal{D}^n, \end{aligned}$$

but the problem is that $(\mathcal{P}_d^*, \Delta_{\mathcal{P}^*})$ does *not* constitute a coalgebra, because we only have that $\Delta_{\mathcal{P}^*} \mathcal{P}_d^* \subseteq (\mathcal{P}_d \otimes \mathcal{P}_d)^*$, yet

Theorem 3.6. $\Delta_{\mathcal{P}^*} \mathcal{P}_d^* \not\subseteq \mathcal{P}_d^* \otimes \mathcal{P}_d^*$.

Proof. At first, let $d = 1$. Consider $D := \sum_{n=0}^{\infty} \mathcal{D}^{n^2} \in \mathcal{P}_1^*$. We have

$$\Delta_{\mathcal{P}^*} D = \sum_{n=0}^{\infty} \sum_{k \leq n^2} \mathcal{D}^k \otimes \mathcal{D}^{n^2-k} = \sum_{k=0}^{\infty} \mathcal{D}^k \otimes \left(\sum_{n^2 \geq k} \mathcal{D}^{n^2-k} \right). \quad (3.7)$$

Assume $\Delta_{\mathcal{P}^*} D \in \mathcal{P}_1^* \otimes \mathcal{P}_1^*$. Then, there is $l \in \mathbb{N}$, $(a_k^i)_{k,i} \subset \mathbb{R}$ and $(L_i)_i \subset \mathcal{P}_1^*$ such that

$$\Delta_{\mathcal{P}^*} D = \sum_{i=1}^l \left(\left(\sum_{k=0}^{\infty} a_k^i \mathcal{D}^k \right) \otimes L_i \right) = \sum_{k=0}^{\infty} \mathcal{D}^k \otimes \left(\sum_{i=1}^l a_k^i L_i \right).$$

This is a contradiction to (3.7) as

$$\dim \text{span} \left\{ \sum_{i=1}^l a_k^i L_i, k \in \mathbb{N}_0 \right\} \leq l, \quad \text{but} \quad \dim \text{span} \left\{ \sum_{n^2 \geq k} \mathcal{D}^{n^2-k}, k \in \mathbb{N}_0 \right\} = \infty.$$

For $d > 1$, considering $D := \sum_{n \in \mathbb{N}_0} \mathcal{D}^{n^2 e_1} \in \mathcal{P}_d^*$ will lead to the same result. \square

However, this is not a big issue for us, since in Section 3.3.1, we will see that we can nevertheless work with the extended operation $\Delta_{\mathcal{P}^*}$ basically as usual, and that the defining identities of a Hopf algebra remain valid for formal series.

3.2 Tensor Hopf algebras

3.2.1 Overview

In this section, we want to introduce the following two structures and show that they are Hopf algebras:

$$\begin{aligned} \mathbb{T}(V) &= \bigoplus_{n=0}^{\infty} V^{\otimes n} \\ \dot{\otimes} : \mathbb{T}(V) \times \mathbb{T}(V) &\rightarrow \mathbb{T}(V) \\ \mathbf{u} : K &\rightarrow \mathbb{T}(V), \quad \mathbf{u}(r) = r \mathbf{1} \\ \Delta_{\mathbb{W}} : \mathbb{T}(V) &\rightarrow \mathbb{T}(V) \otimes \mathbb{T}(V), \quad u = v_1 \dot{\otimes} \dots \dot{\otimes} v_n \text{ is mapped to} \\ \Delta_{\mathbb{W}} u &= \mathbf{1} \otimes u + u \otimes \mathbf{1} + \sum_{0 < i < n} \sum_{\sigma \in \text{Sh}(i, n)} v_{\sigma(1)} \dot{\otimes} \dots \dot{\otimes} v_{\sigma(i)} \otimes v_{\sigma(i+1)} \dot{\otimes} \dots \dot{\otimes} v_{\sigma(n)} \\ \varepsilon : \mathbb{T}(V) &\rightarrow K, \quad \varepsilon(\mathbf{1}) = 1, \quad \varepsilon(V^{\otimes n}) = \{0\} \text{ for } n \in \mathbb{N} \\ \mathcal{S}_{\otimes}^{\mathbb{W}} : \mathbb{T}(V) &\rightarrow \mathbb{T}(V), \quad \mathcal{S}_{\otimes}^{\mathbb{W}}(v_1 \dot{\otimes} \dots \dot{\otimes} v_n) = (-1)^n v_n \dot{\otimes} \dots \dot{\otimes} v_1 \end{aligned}$$

$$\begin{aligned}
& \mathbb{T}(\tilde{V}) \\
& \sqcup : \mathbb{T}(\tilde{V}) \times \mathbb{T}(\tilde{V}) \rightarrow \mathbb{T}(\tilde{V}), \\
& (\tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_i) \sqcup (\tilde{v}_{i+1} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n) := \sum_{\sigma \in \text{Sh}(i,n)} \tilde{v}_{\sigma^{-1}(1)} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_{\sigma^{-1}(n)} \\
& \mathbf{u} : K \rightarrow \mathbb{T}(\tilde{V}), \quad \mathbf{u}(r) = r\mathbf{1} \\
& \Delta_{\dot{\otimes}} : \mathbb{T}(\tilde{V}) \rightarrow \mathbb{T}(\tilde{V}) \otimes \mathbb{T}(\tilde{V}), \\
& \Delta_{\dot{\otimes}}(\tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n) = \sum_{i=0}^n \tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_i \otimes \tilde{v}_{i+1} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n \\
& \varepsilon : \mathbb{T}(\tilde{V}) \rightarrow K, \quad \varepsilon(\mathbf{1}) = 1, \quad \varepsilon(\tilde{V}^{\dot{\otimes} n}) = \{0\} \text{ for } n \in \mathbb{N} \\
& \mathcal{S}_{\sqcup}^{\dot{\otimes}} : \mathbb{T}(\tilde{V}) \rightarrow \mathbb{T}(\tilde{V}), \quad \mathcal{S}_{\sqcup}^{\dot{\otimes}}(v_1 \dot{\otimes} \cdots \dot{\otimes} v_n) = (-1)^n v_n \dot{\otimes} \cdots \dot{\otimes} v_1
\end{aligned}$$

3.2.2 Concatenation product Hopf algebra

Definition 3.7. (Section I.2.2. [Man06], Chapter 4 [HK14]) For some real or complex vector space V and tensor products $\dot{\otimes}$, the *tensor algebra of V* is given by

$$\mathbb{T}(V) := \bigoplus_{n=0}^{\infty} V^{\dot{\otimes} n}, \quad (3.8)$$

where $V^{\dot{\otimes} 0} := \text{span}\{\mathbf{1}\}$ is isomorphic to \mathbb{R} or \mathbb{C} , respectively.

We consider $\dot{\otimes} : \mathbb{T}(V) \times \mathbb{T}(V) \rightarrow \mathbb{T}(V)$ as the algebra product on $\mathbb{T}(V)$ with unit $\mathbf{1}$, also called *concatenation*, while $\mathbb{T}(V) \otimes \mathbb{T}(V) \not\subseteq \mathbb{T}(V)$ reoccurs as the space to which the coproduct maps.

Lemma 3.8. (Section I.2.2. [Man06]) *The couple $(\mathbb{T}(V), \dot{\otimes})$ is a connected graded unital algebra under the grading given by (3.8).*

Proof. Tensor products are bilinear and associativity is fulfilled due to the usual identification $u_1 \dot{\otimes} u_2 \dot{\otimes} u_3 := (u_1 \dot{\otimes} u_2) \dot{\otimes} u_3 = u_1 \dot{\otimes} (u_2 \dot{\otimes} u_3)$. The unit is by definition given by $\mathbf{1}$. \square

Definition 3.9. (Based on the definition of the shuffle product in Chapter 4 [HK14]) Let $\Delta_{\sqcup} : \mathbb{T}(V) \rightarrow \mathbb{T}(V) \otimes \mathbb{T}(V)$ be the linear map generated by $\Delta_{\sqcup}\mathbf{1} := \mathbf{1} \otimes \mathbf{1}$ and

$$\Delta_{\sqcup}u := \mathbf{1} \otimes u + u \otimes \mathbf{1} + \sum_{0 < i < n} \sum_{\sigma \in \text{Sh}(i,n)} v_{\sigma(1)} \dot{\otimes} \cdots \dot{\otimes} v_{\sigma(i)} \otimes v_{\sigma(i+1)} \dot{\otimes} \cdots \dot{\otimes} v_{\sigma(n)} \quad (3.9)$$

for all $u = v_1 \dot{\otimes} \cdots \dot{\otimes} v_n \in V^{\dot{\otimes} n}$, where

$$\text{Sh}(i, n) := \{\sigma \in S_n \mid \sigma(1) < \cdots < \sigma(i), \sigma(i+1) < \cdots < \sigma(n)\}$$

and S_n is the group of permutations of the set $\{1, \dots, n\}$.

Example 3.10. Let $a, b, c, d \in V$ be arbitrary. Since

$$\begin{aligned}
\text{Sh}(1, 4) &= \{(1234), (2134), (3124), (4123)\}, \\
\text{Sh}(2, 4) &= \{(1234), (1324), (1423), (2314), (2413), (3412)\}, \\
\text{Sh}(3, 4) &= \{(1234), (1243), (1342), (2341)\},
\end{aligned}$$

we get

$$\begin{aligned} \Delta_{\omega}(a \dot{\otimes} b \dot{\otimes} c \dot{\otimes} d) &= \mathbf{1} \otimes a \dot{\otimes} b \dot{\otimes} c \dot{\otimes} d + a \dot{\otimes} b \dot{\otimes} c \dot{\otimes} d \otimes \mathbf{1} + a \otimes b \dot{\otimes} c \dot{\otimes} d + b \otimes a \dot{\otimes} c \dot{\otimes} d \\ &\quad + c \otimes a \dot{\otimes} b \dot{\otimes} d + d \otimes a \dot{\otimes} b \dot{\otimes} c + a \dot{\otimes} b \otimes c \dot{\otimes} d + a \dot{\otimes} c \otimes b \dot{\otimes} d \\ &\quad + a \dot{\otimes} d \otimes b \dot{\otimes} c + b \dot{\otimes} c \otimes a \dot{\otimes} d + b \dot{\otimes} d \otimes a \dot{\otimes} c + c \dot{\otimes} d \otimes a \dot{\otimes} b \\ &\quad + a \dot{\otimes} b \dot{\otimes} c \otimes d + a \dot{\otimes} b \dot{\otimes} d \otimes c + a \dot{\otimes} c \dot{\otimes} d \otimes b + b \dot{\otimes} c \dot{\otimes} d \otimes a. \end{aligned}$$

If $b = c$ and $a = d$, we thus have

$$\begin{aligned} \Delta_{\omega}(a \dot{\otimes} b \dot{\otimes} b \dot{\otimes} a) &= \mathbf{1} \otimes a \dot{\otimes} b \dot{\otimes} b \dot{\otimes} a + a \dot{\otimes} b \dot{\otimes} b \dot{\otimes} a \otimes \mathbf{1} + a \otimes b \dot{\otimes} b \dot{\otimes} a + 2b \otimes a \dot{\otimes} b \dot{\otimes} a \\ &\quad + a \otimes a \dot{\otimes} b \dot{\otimes} b + 2a \dot{\otimes} b \otimes b \dot{\otimes} a + a \dot{\otimes} a \otimes b \dot{\otimes} b + b \dot{\otimes} b \otimes a \dot{\otimes} a \\ &\quad + 2b \dot{\otimes} a \otimes a \dot{\otimes} b + a \dot{\otimes} b \dot{\otimes} b \otimes a + 2a \dot{\otimes} b \dot{\otimes} a \otimes b + b \dot{\otimes} b \dot{\otimes} a \otimes a. \end{aligned}$$

Lemma 3.11. (Proposition 1.9 [Reu93], Section I.6.2. [Man06]) *The operator Δ_{ω} is the unique $(\mathbb{T}(V), \dot{\otimes})$ homomorphism to $\mathbb{T}(V) \otimes \mathbb{T}(V)$ satisfying*

$$\Delta_{\omega}v = \mathbf{1} \otimes v + v \otimes \mathbf{1} \quad (3.10)$$

for all $v \in V$. The object $(\mathbb{T}(V), \dot{\otimes}, \Delta_{\omega})$ is a connected graded bialgebra.

Proof. The property (3.10) is included in (3.9), because the sum over $0 < i < n$ is empty for $n = 1$. Let $n \in \mathbb{N}$ and $v_1, \dots, v_{n+1} \in V$ be arbitrary. Since for all $\sigma \in \text{Sh}(i, n+1)$ we have $\sigma(i) = n+1$ or $\sigma(n+1) = n+1$, we get

$$\begin{aligned} \tilde{\Delta}_{\omega}(u_n \dot{\otimes} v_{n+1}) &= \sum_{(u_n)} ((u'_1 \dot{\otimes} v_{n+1}) \otimes u''_n + u'_1 \otimes (u''_2 \dot{\otimes} v_{n+1})) + u_n \dot{\otimes} v_{n+1} + v_{n+1} \dot{\otimes} u_n \\ &= \tilde{\Delta}_{\omega}u_n \dot{\otimes} (v_{n+1} \otimes \mathbf{1} + \mathbf{1} \otimes v_{n+1}) + u_n \dot{\otimes} v_{n+1} + v_{n+1} \dot{\otimes} u_n \\ &= \Delta_{\omega}u_n \dot{\otimes} \Delta_{\omega}v_{n+1} - u_n \dot{\otimes} v_{n+1} \otimes \mathbf{1} - \mathbf{1} \otimes u_n \dot{\otimes} v_{n+1}. \end{aligned}$$

Hence, inductively we prove

$$\Delta_{\omega}(v_1 \dot{\otimes} \dots \dot{\otimes} v_n) = \Delta_{\omega}v_1 \dot{\otimes} \dots \dot{\otimes} \Delta_{\omega}v_n \quad (3.11)$$

for all $n \in \mathbb{N}$ and $v_1, \dots, v_n \in V$, showing that Δ_{ω} is an algebra homomorphism. Since (3.11) is a necessary condition for an algebra homomorphism, Δ_{ω} is indeed the unique algebra homomorphism such that (3.10) holds. It is then also a coproduct due to Lemma 2.11, since $(\text{id} \otimes \Delta_{\omega})\Delta_{\omega}v = (\Delta_{\omega} \otimes \text{id})\Delta_{\omega}v$ holds for all $v \in V$. Together with the counit given by $\varepsilon_{\omega}(\mathbf{1}) = 1$, $\varepsilon_{\omega}(V^{\dot{\otimes} n}) = 0$, $n \geq 1$, all the compatibility requirements are fulfilled, hence $(\mathbb{T}(V), \dot{\otimes}, \Delta_{\omega})$ is a bialgebra. Since from (3.9) we see that

$$\Delta_{\omega}V^{\dot{\otimes} n} \subseteq \bigoplus_{0 \leq m \leq n} V^{\dot{\otimes} m} \otimes V^{\dot{\otimes} n-m},$$

it is also connected graded. □

Lemma 3.12. (Lemma 1.5 [Reu93], Section I.6.2. [Man06]) *The linear operator $\mathcal{S}_{\otimes}^{\omega} : \mathbb{T}(V) \rightarrow \mathbb{T}(V)$ generated by*

$$\mathcal{S}_{\otimes}^{\omega}\mathbf{1} := \mathbf{1}, \quad \mathcal{S}_{\otimes}^{\omega}(v_1 \dot{\otimes} \dots \dot{\otimes} v_n) := (-1)^n v_n \dot{\otimes} \dots \dot{\otimes} v_1 \quad \forall v_i \in V$$

is an antipode of $(\mathbb{T}(V), \dot{\otimes}, \Delta_{\omega})$.

Proof. (Proof of Lemma 1.5 [Reu93]) As $m_{\otimes}(\text{id} \otimes \mathcal{S}_{\otimes}^{\mathbb{w}})\Delta_{\mathbb{w}}\mathbf{1} = m_{\otimes}(\mathcal{S}_{\otimes}^{\mathbb{w}} \otimes \text{id})\Delta_{\mathbb{w}}\mathbf{1} = u\varepsilon\mathbf{1} = \mathbf{1}$, it only remains to show

$$m_{\otimes}(\text{id} \otimes \mathcal{S}_{\otimes}^{\mathbb{w}})\Delta_{\mathbb{w}}u = m_{\otimes}(\mathcal{S}_{\otimes}^{\mathbb{w}} \otimes \text{id})\Delta_{\mathbb{w}}u = 0 \quad \forall u \in V^{\otimes m}, m \in \mathbb{N}, \quad (3.12)$$

where we proceed by induction. Regarding the case $m = 1$, we obtain

$$m_{\otimes}(\text{id} \otimes \mathcal{S}_{\otimes}^{\mathbb{w}})\Delta_{\mathbb{w}}v = \mathbf{1} \dot{\otimes} \mathcal{S}_{\otimes}^{\mathbb{w}}v + v \dot{\otimes} \mathcal{S}_{\otimes}^{\mathbb{w}}\mathbf{1} = -v + v \dot{\otimes} \mathbf{1} = 0$$

as well as

$$m_{\otimes}(\mathcal{S}_{\otimes}^{\mathbb{w}} \otimes \text{id})\Delta_{\mathbb{w}}v = \mathcal{S}_{\otimes}^{\mathbb{w}}\mathbf{1} \dot{\otimes} v + \mathcal{S}_{\otimes}^{\mathbb{w}}v \dot{\otimes} \mathbf{1} = \mathbf{1} \dot{\otimes} v - v = 0.$$

for all $v \in V$. Assuming that (3.12) holds for $m = n$, we have

$$\sum_{(w)} w' \dot{\otimes} \mathcal{S}_{\otimes}^{\mathbb{w}}w'' = \sum_{(w)} \mathcal{S}_{\otimes}^{\mathbb{w}}w' \dot{\otimes} w'' = 0$$

for arbitrary $w \in V^{\otimes n}$. Let $x \in V$ be arbitrary as well. Then,

$$\Delta_{\mathbb{w}}(w \dot{\otimes} x) = \Delta_{\mathbb{w}}w \dot{\otimes} \Delta_{\mathbb{w}}x = \sum_{(w)} (w' \otimes (w'' \dot{\otimes} x) + (w' \dot{\otimes} x) \otimes w'')$$

and therefore

$$\begin{aligned} m_{\otimes}(\mathcal{S}_{\otimes}^{\mathbb{w}} \otimes \text{id})\Delta_{\mathbb{w}}(w \dot{\otimes} x) &= \sum_{(w)} (\mathcal{S}_{\otimes}^{\mathbb{w}}w' \dot{\otimes} (w'' \dot{\otimes} x) + \mathcal{S}_{\otimes}^{\mathbb{w}}(w' \dot{\otimes} x) \dot{\otimes} w'') \\ &= \left(\sum_{(w)} \mathcal{S}_{\otimes}^{\mathbb{w}}w' \dot{\otimes} w'' \right) \dot{\otimes} x + x \dot{\otimes} \sum_{(w)} \mathcal{S}_{\otimes}^{\mathbb{w}}w' \dot{\otimes} w'' = 0. \end{aligned}$$

Analogously, we get

$$\begin{aligned} m_{\otimes}(\text{id} \otimes \mathcal{S}_{\otimes}^{\mathbb{w}})\Delta_{\mathbb{w}}(x \dot{\otimes} w) &= \sum_{(w)} (w' \dot{\otimes} \mathcal{S}_{\otimes}^{\mathbb{w}}(x \dot{\otimes} w'') + (x \dot{\otimes} w') \dot{\otimes} \mathcal{S}_{\otimes}^{\mathbb{w}}w'') \\ &= \left(\sum_{(w)} w' \dot{\otimes} \mathcal{S}_{\otimes}^{\mathbb{w}}w'' \right) \dot{\otimes} x + x \dot{\otimes} \sum_{(w)} w' \dot{\otimes} \mathcal{S}_{\otimes}^{\mathbb{w}}w'' = 0. \end{aligned}$$

As $w \in V^{\otimes n}$, $x \in V$ were arbitrary, we get (3.12) for $m = n + 1$ by linearity of $\mathcal{S}_{\otimes}^{\mathbb{w}}$. \square

Corollary 3.13. (Section I.6.2. [Man06]) $(\mathbb{T}(V), \dot{\otimes}, \Delta_{\mathbb{w}}, \mathcal{S}_{\otimes}^{\mathbb{w}})$ is a connected graded Hopf algebra.

3.2.3 Shuffle product Hopf algebra

Theorem 3.14. Let $\langle \cdot, \cdot \rangle$ be a duality pairing of V with some vector space \tilde{V} . Then, the pair $(\mathbb{T}(V), \mathbb{T}(\tilde{V}))$ is a pair of dual vector spaces with duality pairing

$$\left\langle \sum_{n=0}^{\infty} u_n, \sum_{m=0}^{\infty} \tilde{u}_m \right\rangle_{\mathbb{T}} := \sum_{n=0}^{\infty} \langle u_n, \tilde{u}_n \rangle_{\dot{\otimes}_n}, \quad (3.13)$$

where $\langle a\mathbf{1}, b\mathbf{1} \rangle_{\dot{\otimes}_0} := ab$, $\langle v, \tilde{v} \rangle_{\dot{\otimes}_1} := \langle v, \tilde{v} \rangle$ and

$$\left\langle \sum_{i=1}^k v_1^i \dot{\otimes} \cdots \dot{\otimes} v_n^i, \sum_{j=1}^l \tilde{v}_1^j \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n^j \right\rangle_{\dot{\otimes}_n} := \sum_{i=1}^k \sum_{j=1}^l \langle v_1^i, \tilde{v}_1^j \rangle \cdots \langle v_n^i, \tilde{v}_n^j \rangle.$$

Proof. By repeatedly applying Theorem 2.23, we see that the pairings $\langle \cdot, \cdot \rangle_{\otimes n}$, $n \in \mathbb{N}_0$ are all well-defined and bilinear. Thus, also $\langle \cdot, \cdot \rangle_{\mathbb{T}}$ is well-defined and bilinear. For some arbitrary $0 \neq u = \sum_{n=0}^{\infty} u_n \in \mathbb{T}(\langle V \rangle)$ with $u_m \in V^{\otimes m}$ for all $m \in \mathbb{N}_0$, there is k such that $u_k \neq 0$. If $k = 0$, we are done as $\langle u, \mathbf{1} \rangle_{\mathbb{T}} \neq 0$. Otherwise, we see through repetitive application of Theorem 2.23 that $(V^{\otimes k}, \tilde{V}^{\otimes k})$ is a pair of dual vector spaces and hence there is $\tilde{u} \in \tilde{V}^{\otimes k}$ such that

$$0 \neq \langle u_k, \tilde{u} \rangle_{\otimes k} = \langle u, \tilde{u} \rangle_{\mathbb{T}}.$$

Analogously, we show that for each $\tilde{x} \in \mathbb{T}(\tilde{V})$ there is $x \in \mathbb{T}(V)$ such that $\langle x, \tilde{x} \rangle_{\mathbb{T}} \neq 0$. \square

Lemma 3.15. (Sections 1.4 and 1.5 [Reu93], Chapter 4 [HK14]) *The shuffle product $\sqcup : \mathbb{T}(\tilde{V}) \times \mathbb{T}(\tilde{V}) \rightarrow \mathbb{T}(\tilde{V})$ bilinearly generated by*

$$\mathbf{1} \sqcup u := u \sqcup \mathbf{1} := u$$

and

$$(\tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_i) \sqcup (\tilde{v}_{i+1} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n) := \sum_{\sigma \in \text{Sh}(i, n)} \tilde{v}_{\sigma^{-1}(1)} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_{\sigma^{-1}(n)}$$

is dual to Δ_{\sqcup} .

Proof. Let $v_1, \dots, v_n \in V$ and $\tilde{v}_1, \dots, \tilde{v}_{k+l} \in \tilde{V}$ be arbitrary. Put $w := v_1 \dot{\otimes} \cdots \dot{\otimes} v_n$, $a := \tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_k$ and $b := \tilde{v}_{k+1} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_{k+l}$. If $k+l \neq n$, then $\langle w, a \sqcup b \rangle_{\mathbb{T}} = 0 = \langle \Delta_{\sqcup} w, a \otimes b \rangle_{\mathbb{T} \otimes}$ as

$$w \in V^{\otimes n}, \quad a \sqcup b \in \tilde{V}^{\otimes(k+l)}, \quad \Delta_{\sqcup} w \in \bigoplus_{i=0}^n V^{\otimes i} \otimes V^{\otimes(n-i)}, \quad a \otimes b \in V^{\otimes k} \otimes V^{\otimes l}. \quad (3.14)$$

If $n = k+l$, then

$$\begin{aligned} \langle w, a \sqcup b \rangle_{\mathbb{T}} &= \sum_{\sigma \in \text{Sh}(k, n)} \prod_{j=1}^n \langle v_j, \tilde{v}_{\sigma^{-1}(j)} \rangle = \sum_{\sigma \in \text{Sh}(k, n)} \prod_{j=1}^n \langle v_{\sigma(j)}, \tilde{v}_j \rangle \\ &= \sum_{\sigma \in \text{Sh}(k, n)} \langle v_{\sigma(1)} \dot{\otimes} \cdots \dot{\otimes} v_{\sigma(k)}, \tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_k \rangle_{\otimes k} \langle v_{\sigma(k+1)} \dot{\otimes} \cdots \dot{\otimes} v_{\sigma(n)}, \tilde{v}_{k+1} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n \rangle_{\otimes l} \\ &= \langle \Delta_{\sqcup} w, a \otimes b \rangle_{\mathbb{T} \otimes}. \end{aligned}$$

For $z \in \mathbb{T}(V)$ we have

$$\langle z, \mathbf{1} \sqcup u \rangle_{\mathbb{T}} = \langle z, u \sqcup \mathbf{1} \rangle_{\mathbb{T}} = \langle z, u \rangle_{\mathbb{T}} = \langle \Delta_{\sqcup} z, \mathbf{1} \otimes u \rangle_{\mathbb{T} \otimes} = \langle \Delta_{\sqcup} z, u \otimes \mathbf{1} \rangle_{\mathbb{T} \otimes}$$

and for $x, y \in \mathbb{T}(\tilde{V})$ also

$$\langle \mathbf{1}, x \sqcup y \rangle_{\mathbb{T}} = \langle \mathbf{1}, x \rangle_{\mathbb{T}} \langle \mathbf{1}, y \rangle_{\mathbb{T}} = \langle \mathbf{1} \otimes \mathbf{1}, x \otimes y \rangle_{\mathbb{T} \otimes} = \langle \Delta_{\sqcup} \mathbf{1}, a \otimes b \rangle_{\mathbb{T} \otimes}.$$

The duality of \sqcup and Δ_{\sqcup} then follows by (bi)linearity. \square

Lemma 3.16. (Chapter 4 [HK14]) *The deconcatenation $\Delta_{\otimes} : \mathbb{T}(\tilde{V}) \rightarrow \mathbb{T}(\tilde{V}) \otimes \mathbb{T}(\tilde{V})$ linearly generated by*

$$\Delta_{\otimes} \mathbf{1} := \mathbf{1} \otimes \mathbf{1}, \quad \Delta_{\otimes}(\tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n) = \sum_{i=0}^n \tilde{v}_1 \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_i \otimes \tilde{v}_{i+1} \dot{\otimes} \cdots \dot{\otimes} \tilde{v}_n \quad (3.15)$$

is dual to the concatenation $\dot{\otimes}$.

Proof. Consider $w = \tilde{v}_1 \dot{\otimes} \dots \dot{\otimes} \tilde{v}_n \in \tilde{V}$ and $a = v_1 \dot{\otimes} \dots \dot{\otimes} v_k, b = v_{k+1} \dot{\otimes} \dots \dot{\otimes} v_{k+l} \in V, 0 < k < n$. Then, $\langle a \otimes b, \Delta_{\otimes} w \rangle_{T \otimes} = 0 = \langle a \dot{\otimes} b, w \rangle_T$ if $k+l \neq n$ (cf. (3.14)). If $k+l = n$, we have

$$\begin{aligned} \langle a \otimes b, \Delta_{\otimes} w \rangle_{T \otimes} &= \langle v_1 \dot{\otimes} \dots \dot{\otimes} v_k, \tilde{v}_1 \dot{\otimes} \dots \dot{\otimes} \tilde{v}_k \rangle_{\otimes k} \langle v_{k+1} \dot{\otimes} \dots \dot{\otimes} v_n, \tilde{v}_{k+1} \dot{\otimes} \dots \dot{\otimes} \tilde{v}_n \rangle_{\otimes(n-k)} \\ &= \langle a \dot{\otimes} b, w \rangle_T. \end{aligned}$$

The remainder of the proof is completely analogous to that of Lemma 3.15. \square

Lemma 3.17. *The linear operator $\mathcal{S}_{\mathbb{W}}^{\otimes} : T(\tilde{V}) \rightarrow T(\tilde{V})$ generated by*

$$\mathcal{S}_{\mathbb{W}}^{\otimes} \mathbf{1} := \mathbf{1}, \quad \mathcal{S}_{\mathbb{W}}^{\otimes}(\tilde{v}_1 \dot{\otimes} \dots \dot{\otimes} \tilde{v}_n) := (-1)^n \tilde{v}_n \dot{\otimes} \dots \dot{\otimes} \tilde{v}_1$$

is dual to $\mathcal{S}_{\mathbb{W}}^{\mathbb{W}}$.

Proof. We have for $u \in V^{\otimes m}$ and $v \in V^{\otimes n}$ that $\langle u, \mathcal{S}_{\mathbb{W}}^{\otimes} w \rangle_T = 0 = \langle \mathcal{S}_{\mathbb{W}}^{\mathbb{W}} u, w \rangle_T$ if $m \neq n$. Also, $\langle \mathcal{S}_{\mathbb{W}}^{\mathbb{W}} \mathbf{1}, \mathbf{1} \rangle_T = 1 = \langle \mathbf{1}, \mathcal{S}_{\mathbb{W}}^{\otimes} \mathbf{1} \rangle_T$ and for $u = v_1 \dot{\otimes} \dots \dot{\otimes} v_n, w = \tilde{v}_1 \dot{\otimes} \dots \dot{\otimes} \tilde{v}_n$ we get

$$\langle u, \mathcal{S}_{\mathbb{W}}^{\otimes} w \rangle_T = (-1)^n \prod_{i=1}^n \langle v_i, \tilde{v}_{n-i} \rangle = \langle \mathcal{S}_{\mathbb{W}}^{\mathbb{W}} u, w \rangle_T.$$

\square

Theorem 3.18. (Proposition 1.9 [Reu93], Chapter 4 [HK14]) $(T(\tilde{V}), \sqcup, \Delta_{\otimes}, \mathcal{S}_{\mathbb{W}}^{\otimes})$ is a connected graded Hopf algebra and dual to the Hopf algebra $(T(V), \dot{\otimes}, \Delta_{\mathbb{W}}, \mathcal{S}_{\mathbb{W}}^{\mathbb{W}})$.

Proof. The operations \sqcup, Δ_{\otimes} and $\mathcal{S}_{\mathbb{W}}^{\otimes}$ obviously are compatible with the grading. The fact that $(T(\tilde{V}), \sqcup, \Delta_{\otimes}, \mathcal{S}_{\mathbb{W}}^{\otimes})$ follows from Theorem 2.26 applied to the duality relations of Lemmas 3.15, 3.16 and 3.17 and the obvious dualities of unitmaps and counits. \square

3.3 Formal series, Lie algebra and truncations

3.3.1 Formal series

This section is based on the use of formal series and infinite linearity in [Reu93], though our definitions are a bit different.

For any vector space G graded by $(G_i)_i$, i.e. $G = \bigoplus_{i=0}^{\infty} G_i$ for subspaces G_i , the corresponding space of formal series is given by

$$G_{\Pi} := \prod_{i=0}^{\infty} G_i := \left\{ \sum_{n=0}^{\infty} u_n \mid u_m \in G_m \forall m \in \mathbb{N}_0 \right\},$$

where the product is the Cartesian product of vector spaces (Chapter V Section 5. [RR64]). For the space $G \otimes G$, consider the induced grading

$$([G \otimes G]_i)_i := \left(\bigoplus_{m \leq i} G_m \otimes G_{i-m} \right)_i. \quad (3.16)$$

Let $\pi_n : G_{\Pi} \rightarrow G_n$ and $\underline{\pi}_n : [G \otimes G]_{\Pi} \rightarrow [G \otimes G]_n$ be the canonical projections. Generally, for $k \in \mathbb{N}, k \geq 2$, the grading of the space $G^{\otimes k}$ is given by

$$([G^{\otimes k}]_i)_i := \left(\bigoplus_{\substack{m_1, \dots, m_k \in \mathbb{N}_0: \\ m_1 + \dots + m_k = i}} G_{m_1} \otimes \dots \otimes G_{m_k} \right)_i,$$

and we denote the canonical projection on $[G^{\otimes k}]_n$ by $\pi_n^{[k]}$.

Theorem 3.19.

- (i) If (G, \diamond) is a graded algebra, then G_{Π} is an algebra together with the extended product $\diamond_x : G_{\Pi} \times G_{\Pi} \rightarrow G_{\Pi}$ given by

$$\pi_n m_{\diamond_x}(a \otimes b) = \pi_n(a \diamond_x b) := \sum_{i=0}^n \pi_i a \diamond \pi_{n-i} b = m_{\diamond_x} \pi_n(a \otimes b).$$

- (ii) If (G, Δ) is a graded coalgebra, then the extended coproduct $\Delta_x : G_{\Pi} \rightarrow [G \otimes G]_{\Pi}$ given by

$$\pi_n \Delta_x := \Delta \pi_n$$

is coassociative. If (G, Δ) is counitary with counit ε , then $\varepsilon_x : G_{\Pi} \rightarrow K$ given by

$$\varepsilon_x := \varepsilon \pi_0.$$

fulfills the counit property

$$(\varepsilon_x \otimes \text{id}) \Delta_x = (\text{id} \otimes \varepsilon_x) \Delta_x = \text{id}.$$

- (iii) If (G, \diamond, Δ) is a graded bialgebra with unit map u and counit ε over the field K such that $B^{>0} \subseteq \ker \varepsilon$, then $(G_{\Pi}, \diamond_x, \Delta_x)$ fulfills the compatibility requirements

(a) $\Delta_x m_{\diamond_x} = (m_{\diamond_x} \otimes m_{\diamond_x})(\text{id} \otimes \hat{\tau} \otimes \text{id})(\Delta_x \otimes \Delta_x)$ (Δ_x is an algebra homomorphism from (G_{Π}, \diamond_x) to $([G \otimes G]_{\Pi}, \diamond_x)$),

(b) $\Delta_x u = u \otimes u$,

(c) $\varepsilon_x m_{\diamond_x} = \varepsilon_x \otimes \varepsilon_x$ (ε_x is an algebra homomorphism from (G_{Π}, \diamond_x) to (K, \cdot)),

where \diamond_x is defined as in (2.9).

- (iv) If $(G, \diamond, \Delta, \mathcal{S})$ is a graded Hopf algebra such that $B^{>0} \subseteq \ker \varepsilon$ and $\mathcal{S}_x : G_{\Pi} \rightarrow G_{\Pi}$ is the extended antipode given by

$$\pi_n \mathcal{S}_x := \mathcal{S} \pi_n,$$

then \mathcal{S}_x fulfills the antipode property

$$m_{\diamond_x}(\mathcal{S}_x \otimes \text{id}) \Delta_x = m_{\diamond_x}(\text{id} \otimes \mathcal{S}_x) \Delta_x = u \varepsilon_x.$$

Proof.

- (i) From associativity of \diamond , we conclude

$$\begin{aligned} \pi_n((a \diamond_x b) \diamond_x c) &= \sum_{i=0}^n \pi_i(a \diamond_x b) \diamond \pi_{n-i} c = \sum_{i=0}^n \sum_{j=0}^i (\pi_j a \diamond \pi_{i-j} b) \diamond \pi_{n-i} c \\ &= \sum_{i=0}^n \sum_{j=0}^i \pi_i a \diamond (\pi_{i-j} b \diamond \pi_{n-i} c) = \sum_{k=0}^n \sum_{l=0}^{n-k} \pi_k a \diamond (\pi_l b \diamond \pi_{n-k-l} c) \\ &= \sum_{k=0}^n \pi_k a \diamond \pi_{n-k}(b \diamond_x c) = \pi_n(a \diamond_x (b \diamond_x c)). \end{aligned}$$

(ii) Coassociativity of Δ implies

$$\begin{aligned}\pi_n^{[3]}(\Delta_x \otimes \text{id})\Delta_x &= (\Delta \otimes \text{id})\pi_n\Delta_x = (\Delta \otimes \text{id})\Delta\pi_n = (\text{id} \otimes \Delta)\Delta\pi_n = (\text{id} \otimes \Delta)\pi_n\Delta_x \\ &= \pi_n^{[3]}(\text{id} \otimes \Delta_x)\Delta_x.\end{aligned}$$

(iii) Due to the first compatibility requirement of Definition 2.8, we have

$$\begin{aligned}\pi_n\Delta_x m_{\circ_x} &= \Delta\pi_n m_{\circ_x} = \Delta m_{\circ} \pi_n = (m_{\circ} \otimes m_{\circ})\hat{\tau}_{1324}(\Delta \otimes \Delta)\pi_n = (m_{\circ} \otimes m_{\circ})\hat{\tau}_{1324}\pi_n^{[4]}(\Delta_x \otimes \Delta_x) \\ &= (m_{\circ} \otimes m_{\circ})\pi_n^{[4]}\hat{\tau}_{1324}(\Delta_x \otimes \Delta_x) = \pi_n(m_{\circ_x} \otimes m_{\circ_x})\hat{\tau}_{1324}(\Delta_x \otimes \Delta_x)\end{aligned}$$

The second compatibility requirement follows from

$$\Delta_x \mathbf{1} = \Delta_x \pi_0 \mathbf{1} = \pi_0 \Delta \mathbf{1} = \pi_0 (\mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1}.$$

Finally, from the original third compatibility requirement of Definition 2.8, we conclude

$$\varepsilon_x m_{\circ_x} = \varepsilon \pi_0 m_{\circ_x} = \varepsilon m_{\circ} \pi_0 = (\varepsilon \otimes \varepsilon)\pi_0 = \varepsilon_x \otimes \varepsilon_x.$$

(iv) From the original antipode property (2.14), it follows

$$\begin{aligned}\pi_n m_{\circ_x} (\text{id} \otimes \mathcal{S}_x)\Delta_x &= m_{\circ} \pi_n (\text{id} \otimes \mathcal{S}_x) = m_{\circ} (\text{id} \otimes \mathcal{S})\pi_n \Delta_x = m_{\circ} (\text{id} \otimes \mathcal{S})\Delta\pi_n = u\varepsilon\pi_n = \pi_n u\varepsilon_x, \\ \text{since } u\varepsilon\pi_n &= 0 = \pi_n u\varepsilon_x \text{ if } n \neq 0 \text{ and otherwise } u\varepsilon\pi_0 = u\varepsilon_x = \pi_0 u\varepsilon_x. \text{ Likewise, } \pi_n m_{\circ_x} (\mathcal{S}_x \otimes \text{id})\Delta_x = \pi_n u\varepsilon_x.\end{aligned}$$

□

In the following, we will simply also write $\diamond, \Delta, \varepsilon, \mathcal{S}$ for $\diamond_x, \Delta_x, \varepsilon_x, \mathcal{S}_x$ when it is clear from the context that we mean the extended operations on G_{Π} .

3.3.2 Lie Algebras

Definition 3.20. (Section 0.1 [Reu93]) A *Lie algebra* is a vector space \mathfrak{l} together with a bilinear map $\mathfrak{b} = [\cdot, \cdot] : \mathfrak{l} \times \mathfrak{l} \rightarrow \mathfrak{l}$, the *Lie bracket*, with the properties

1. $[x, x] = 0$ for all $x \in \mathfrak{l}$,
2. $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all $a, b, c \in \mathfrak{l}$ (*Jacobi's identity*).

Remark 3.21. (Section 0.1 [Reu93]) As for all $a, b \in \mathfrak{l}$ we have

$$0 = [a + b, a + b] = [a, a] + [a, b] + [b, a] + [b, b] = [a, b] + [b, a],$$

the Lie bracket is *antisymmetric*, i.e.

$$[a, b] = -[b, a] \quad \forall a, b \in \mathfrak{l}.$$

Definition 3.22. A sub Lie algebra of a Lie algebra $(\mathfrak{l}, \mathfrak{b})$ is a subspace $V \subseteq \mathfrak{l}$ such that

$$[V, V] \subseteq V.$$

The sub Lie algebra generated by a subspace $S \subseteq \mathfrak{l}$, i.e. the intersection of all sub Lie algebras containing S , is denoted by $\langle S \rangle_{\mathfrak{b}}$.

Remark 3.23. (Section 0.1 [Reu93]) For any algebra (A, \diamond) , there is a Lie algebra $(A, \mathfrak{b}_\diamond)$ given by

$$\mathfrak{b}_\diamond(a, b) := [a, b]_\diamond := a \diamond b - b \diamond a \quad \forall a, b \in A.$$

If A is an algebra graded by $(A_i)_i$, put $\mathfrak{g}_A := \prod_{i=1}^{\infty} A_i$, which is a Lie algebra together with the Lie bracket \mathfrak{b}_\diamond of the extended product \diamond .

Definition 3.24. (Based on Section 1.3 [Reu93]) For a Lie algebra $(\mathfrak{l}, \mathfrak{b})$, the linear operator $\mathfrak{R} : \mathbb{T}(\mathfrak{l}) \rightarrow \mathfrak{l}$ recursively generated by

$$\mathfrak{R}\mathbf{1} = 0, \quad \mathfrak{R}x = x, \quad \mathfrak{R}(x \dot{\otimes} u) = [x, \mathfrak{R}u] \quad \forall x \in \mathfrak{l}, u \in \bigoplus_{n=1}^{\infty} \mathfrak{l}^{\dot{\otimes} n}$$

is called *right norm bracketing*.

Theorem 3.25. (Based on Section 0.4.1 [Reu93]) *For a subspace W of a Lie algebra $(\mathfrak{l}, \mathfrak{b})$, we have*

$$\mathfrak{R}\mathbb{T}(W) = \langle W \rangle_{\mathfrak{b}}.$$

Proof. (' \supseteq ') part of proof is taken from Section 0.4.1 [Reu93]) On the one hand, we have $\mathfrak{R}\mathbb{T}(W) \subseteq \langle W \rangle_{\mathfrak{b}}$, since if $\mathfrak{R}u \in \langle W \rangle_{\mathfrak{b}}$ and $w \in W$, then also $\mathfrak{R}(w \dot{\otimes} u) = [w, \mathfrak{R}u] \in \langle W \rangle_{\mathfrak{b}}$, because $\langle W \rangle_{\mathfrak{b}}$ is a sub Lie algebra containing W . On the other hand, each $v \in \langle W \rangle_{\mathfrak{b}}$ can be written as a linear combination of nested Lie brackets. Since

$$[[a, b], c] = -[[b, c], a] - [[c, a], b] = [a, [b, c]] + [b, [c, a]] = [a, [b, c]] - [b, [a, c]] \quad \forall a, b, c \in \langle W \rangle_{\mathfrak{b}}$$

due to Jacobi's identity and antisymmetry, we can inductively transform all nested Lie brackets into a linear combination of right norm bracketings, and thus we also have $\mathfrak{R}\mathbb{T}(W) \supseteq \langle W \rangle_{\mathfrak{b}}$. \square

3.3.3 Exponential map

Let A be a connectedly graded algebra, this time over a field of characteristic zero. Put $\mathcal{G}_A := \mathbf{1} + \mathfrak{g}_A \subseteq A_{\Pi}$ (Section II.3. [Man06]). Define the maps $\exp_\diamond : \mathfrak{g}_A \rightarrow \mathcal{G}_A$ and $\log_\diamond : \mathcal{G}_A \rightarrow \mathfrak{g}_A$ by

$$\exp_\diamond(u) := \sum_{n=0}^{\infty} \frac{u^{\diamond n}}{n!} \quad \text{and} \quad \log_\diamond(\mathbf{1} + u) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1} u^{\diamond n}}{n} \quad (3.17)$$

(Equations (3.1.1) and (3.1.2) [Reu93], Section II.3. [Man06]).

Theorem 3.26.

1. (Equation (3.1.3) [Reu93], Section II.3. [Man06], Section 2.3 [HK14]) *The maps \exp_\diamond and \log_\diamond are bijective and \log_\diamond is the inverse function of \exp_\diamond .*
2. (Based on the proof of Theorem 3.2 [Reu93]) *If $u_1, u_2 \in \mathfrak{g}_A$ with $u_1 \diamond u_2 = u_2 \diamond u_1$, then*

$$\exp_\diamond(u_1) \diamond \exp_\diamond(u_2) = \exp_\diamond(u_2) \diamond \exp_\diamond(u_1) = \exp_\diamond(u_1 + u_2).$$

If $v_1, v_2 \in \mathcal{G}_A$ with $v_1 \diamond v_2 = v_2 \diamond v_1$, then

$$\log_\diamond(v_1) + \log_\diamond(v_2) = \log_\diamond(v_1 \diamond v_2).$$

3. (Based on Proposition II.3.1. [Man06]) *The pair $(\mathcal{G}_A, \diamond)$ forms a group.*

We do not proof Statement 1. of the theorem. It is a standard fact for the case of the tensor algebra $(T(V), \otimes)$, and the general case follows easily from it through looking at suitable homomorphisms.

Proof of the remainder.

2. Let $u_1, u_2 \in \mathfrak{g}_A$ with $u_1 \diamond u_2 = u_2 \diamond u_1$ be arbitrary. Then,

$$\begin{aligned} \exp_{\circ}(u_1 + u_2) &= \sum_{n=0}^{\infty} \frac{(u_1 + u_2)^{\diamond n}}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \frac{u_1^{\diamond m} \diamond u_2^{\diamond(n-m)}}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{u_1^{\diamond m} \diamond u_2^{\diamond(n-m)}}{m!(n-m)!} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{u_1^{\diamond k} \diamond u_2^{\diamond l}}{k!l!} = \exp_{\circ}(u_1) \diamond \exp_{\circ}(u_2). \end{aligned}$$

Let $v_1, v_2 \in \mathcal{G}_A$ with $v_1 \diamond v_2 = v_2 \diamond v_1$ be arbitrary. Then, we obviously have $\log_{\circ}(v_1) \diamond \log_{\circ}(v_2) = \log_{\circ}(v_2) \diamond \log_{\circ}(v_1)$, and thus

$$\exp_{\circ}(\log_{\circ}(v_1) + \log_{\circ}(v_2)) = \exp_{\circ}(\log_{\circ}(v_1)) \diamond \exp_{\circ}(\log_{\circ}(v_2)) = v_1 \diamond v_2 = \exp_{\circ}(\log_{\circ}(v_1 + v_2)).$$

Due to injectivity of \exp_{\circ} , we get $\log_{\circ}(v_1) + \log_{\circ}(v_2) = \log_{\circ}(v_1 \diamond v_2)$.

3. We have $\mathfrak{g}_A \diamond \mathfrak{g}_A \subseteq \mathfrak{g}_A$ due to the grading, and thus also $\mathcal{G}_A \diamond \mathcal{G}_A \subseteq \mathcal{G}_A$ since $\mathcal{G}_A = \mathbf{1} + \mathfrak{g}_A$. Furthermore, since \exp_{\circ} is surjective, for any $y \in \mathcal{G}_A$, there is $x \in \mathfrak{g}_A$ such that $y = \exp_{\circ}(x)$, and due to 2.,

$$y \diamond \exp_{\circ}(-x) = \exp_{\circ}(x) \diamond \exp_{\circ}(-x) = \exp_{\circ}(x - x) = \exp_{\circ}(0) = \mathbf{1}.$$

□

Theorem 3.27. *Let (A, \diamond) and (\hat{A}, \blacklozenge) be connectedly graded algebras and $\Lambda : A_{\Pi} \rightarrow \hat{A}_{\Pi}$ be an algebra homomorphism such that $\Lambda \mathbf{1} = \mathbf{1}$ and $\Lambda \pi_n v = \pi_n \Lambda v$ for all $v \in A_{\Pi}$, $n \in \mathbb{N}_0$. Then,*

$$\Lambda \exp_{\circ}(u) = \exp_{\bullet}(\Lambda u) \quad \forall u \in \mathfrak{g}_A$$

and

$$\Lambda \log_{\circ}(v) = \log_{\bullet}(\Lambda v) \quad \forall v \in \mathcal{G}_A.$$

Proof. For each $u \in \mathfrak{g}_A$, we have

$$\Lambda \exp_{\circ}(u) = \Lambda \sum_{n=0}^{\infty} \frac{u^{\diamond n}}{n!} = \sum_{n=0}^{\infty} \frac{\Lambda u^{\diamond n}}{n!} = \sum_{n=0}^{\infty} \frac{(\Lambda u)^{\bullet n}}{n!} = \exp_{\bullet}(\Lambda u),$$

were the interchangeability of the infinite sum and Λ comes from the fact that $\Lambda \pi_n v = \pi_n \Lambda v$ for all $v \in A_{\Pi}$, $n \in \mathbb{N}_0$. For each $\mathbf{1} + u \in \mathcal{G}_A$, we have

$$\begin{aligned} \Lambda \log_{\circ}(\mathbf{1} + u) &= \Lambda \sum_{n=1}^{\infty} \frac{(-1)^{n-1} u^{\diamond n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Lambda u^{\diamond n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (\Lambda u)^{\bullet n}}{n} = \log_{\bullet}(\mathbf{1} + \Lambda u) \\ &= \log_{\bullet}(\Lambda(\mathbf{1} + u)). \end{aligned}$$

□

3.3.4 Truncations

This subsection prepares a detailed treatment of the notion of truncations as they are used in Section 2.4 and in Chapter 4 [HK14]. For an algebra (A, \diamond) graded by $(A_i)_i$, the space $A^{>n} := \bigoplus_{m=n+1}^{\infty} A_m$ is a two-sided ideal of A as well as the space $A_{\Pi}^{>n} := \prod_{m=n+1}^{\infty} A_m$ is an ideal of (A_{Π}, \diamond) .

We assume that the following statement is standard, but did not find it in the literature we used. Part of the statement on quotient coalgebras is given in Section III.1 [Kas95], though.

Theorem 3.28. *Let I be a subspace of a vector space V and $\pi : V \rightarrow V/I$ be the quotient map.*

(i) *If (V, \diamond) is an algebra, there is a map $\diamond_{\pi} : V/I \times V/I \rightarrow V/I$ such that*

$$m_{\diamond_{\pi}}(\pi \otimes \pi) = \pi m_{\diamond} \quad (3.18)$$

iff I is a two-sided ideal of (V, \diamond) . In this case, the map is unique and $(V/I, \diamond_{\pi})$ is an algebra, the quotient algebra.

(ii) *If (V, Δ) is a coalgebra, there is a map $\Delta_{\pi} : V/I \rightarrow V/I \otimes V/I$ such that*

$$\Delta_{\pi} \pi = (\pi \otimes \pi) \Delta$$

iff I is a two-sided coideal of (V, Δ) . In this case, the map is unique and $(V/I, \Delta_{\pi})$ is a coalgebra, the quotient coalgebra.

Proof.

(i) If there is such a map \diamond_{π} , then

$$\pi(I \diamond V + V \diamond I) = \pi m_{\diamond}(I \otimes V + V \otimes I) = m_{\diamond_{\pi}}(\pi \otimes \pi)(I \otimes V + V \otimes I) = \{0\},$$

hence $I \diamond V + V \diamond I \in \ker \pi = I$, thus I is a two-sided ideal.

If I is a two-sided ideal, then a map \diamond_{π} is well-defined through

$$(\pi v_1 \diamond_{\pi} \pi v_2) := \pi(v_1 \diamond v_2).$$

Indeed, if $\pi v_1 = \pi w_1$ and $\pi v_2 = \pi w_2$, then

$$v_1 \otimes v_2 - w_1 \otimes w_2 \in \ker(\pi \otimes \pi) = \ker \pi \otimes V + V \otimes \ker \pi = I \otimes V + V \otimes I$$

hence

$$v_1 \diamond v_2 - w_1 \diamond w_2 \in I \diamond V + V \diamond I \subseteq I$$

hence $\pi(v_1 \diamond v_2) = \pi(w_1 \diamond w_2)$.

Uniqueness is clear, since we just showed that the demanded equation (3.18) is sufficient as a definition. The only thing left to show is associativity of \diamond_{π} . If $v_1, v_2, v_3 \in V$, then

$$\begin{aligned} (\pi v_1 \diamond_{\pi} \pi v_2) \diamond_{\pi} \pi v_3 &= \pi(v_1 \diamond v_2) \diamond_{\pi} \pi v_3 = \pi((v_1 \diamond v_2) \diamond v_3) = \pi(v_1 \diamond (v_2 \diamond v_3)) \\ &= \pi v_1 \diamond_{\pi} \pi(v_2 \diamond v_3) = \pi v_1 \diamond_{\pi} (\pi v_2 \diamond_{\pi} \pi v_3). \end{aligned}$$

(ii) Due to the formal duality of algebra products and coproducts, the proof of the second statement is analogous to the proof of the first one. □

Due to the grading, we may identify both $A_{\Pi}/A_{\Pi}^{>n}$ and $A/A^{>n}$ canonically with $A^n := \bigoplus_{m=0}^n A_m$. The quotient map $\pi^n : A_{\Pi} \rightarrow A^n$ then restricts to the other quotient map $\pi^n|_A : A \rightarrow A^n$ and is in both cases just the canonical projection onto A^n . In either case, we obtain the same quotient algebra (A^n, \diamond_n) , which is called the *n-th grade truncation* of the graded algebra A , or shortly a *(n-th grade) truncated algebra*.

On the other hand, starting from a graded coalgebra (C, Δ) graded by $(C_i)_i$, the space $C^{>n}$ is not generally a coideal of C , as well as $C_{\Pi}^{>n}$ does not generally fulfill the coideal property with respect to the extended coalgebra C_{Π} . Fortunately though, C^n is both a subcoalgebra of C and fulfills the subcoalgebra property with respect to the extended coalgebra C_{Π} . We call (C^n, Δ_n) where $\Delta_n := \Delta|_{C^n}$ the *n-th grade truncation* of the graded coalgebra (C, Δ) , or shortly a *(n-th grade) truncated coalgebra*. We will often simply write (C^n, Δ) instead of (C^n, Δ_n) .

In the case of a graded bialgebra, we will combine the notions of truncated algebra and truncated coalgebra to that of a truncated bialgebra.

Theorem 3.29. *Let (B, \diamond, Δ) be a bialgebra graded by $(B_i)_i$ over the field K with unit map u and counit ε . Then its n-th grade truncation $(B^n, \diamond_n, \Delta_n)$, called a (n-th grade) truncated bialgebra, fulfills the truncated compatibility requirements*

1. $\Delta_n m_{\diamond_n} = m_{\Delta_n}(\Delta_n \otimes \Delta_n)$ (Δ_n is an algebra homomorphism from (B^n, \diamond_n) to $([B \otimes B]^n, \diamond_n)$),
2. $\Delta_n u = u \otimes u$ (u is a coalgebra homomorphism from (K, id_K) to (B^n, Δ_n)),
3. $\varepsilon_n m_{\diamond_n} = \varepsilon_n \otimes \varepsilon_n$ (ε_n is an algebra homomorphism from (B^n, \diamond_n) to (K, \cdot)),

where $m_{\diamond_n} : B^n \otimes B^n \rightarrow B^n$ is the linear operator corresponding to \diamond_n , the pair $([B \otimes B]^n, \diamond_n)$ is the truncation of the algebra $(B \otimes B, \diamond)$ defined in (2.9) under the grading $([B \otimes B]_i)_i$ as defined in (3.16) and m_{Δ_n} is the linear operator corresponding to Δ_n .

Proof. Let π^n denote the projection of $B \otimes B$ onto $[B \otimes B]^n$.

$$\begin{aligned} \Delta_n m_{\diamond_n}(\pi^n \otimes \pi^n) &= \Delta_n \pi^n m_{\diamond} = \pi^n \Delta_n m_{\diamond} = \pi^n m_{\Delta}(\Delta \otimes \Delta) = m_{\Delta_n}(\pi^n \otimes \pi^n)(\Delta \otimes \Delta) \\ &= m_{\Delta_n}(\Delta_n \otimes \Delta_n)(\pi^n \otimes \pi^n) \end{aligned}$$

Since $\pi^n \otimes \pi^n$ is a surjective map into $B^n \otimes B^n$, we get $\Delta_n m_{\diamond_n} = m_{\Delta_n}(\Delta_n \otimes \Delta_n)$. The equation $\Delta_n u = u \otimes u$ is clear since obviously $\Delta_n \mathbf{1} = \Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$. Finally,

$$\varepsilon_n m_{\diamond_n}(\pi^n \otimes \pi^n) = \varepsilon_n \pi^n m_{\diamond} = \varepsilon m_{\diamond} = (\varepsilon \otimes \varepsilon) = (\varepsilon_n \otimes \varepsilon_n)(\pi^n \otimes \pi^n).$$

□

Theorem 3.30. *Let $(H, \diamond, \Delta, \mathcal{S})$ be a Hopf algebra graded by $(H_i)_i$ with unit map u and counit ε . Then $\mathcal{S}_n := \mathcal{S}|_{H^n}$ fulfills the antipode property*

$$m_{\diamond_n}(\mathcal{S}_n \otimes \text{id})\Delta_n = m_{\diamond_n}(\text{id} \otimes \mathcal{S}_n)\Delta_n = u\varepsilon_n$$

in its n-th grade truncation $(B^n, \diamond_n, \Delta_n, \mathcal{S}_n)$, which we call a (n-th grade) truncated Hopf algebra.

Proof.

$$\begin{aligned} m_{\diamond_n}(\mathcal{S}_n \otimes \text{id})\Delta_n \pi^n &= m_{\diamond_n}(\mathcal{S}_n \otimes \text{id})\pi^n \Delta = m_{\diamond_n} \pi^n (\mathcal{S} \otimes \text{id})\Delta = m_{\diamond_n}(\pi^n \otimes \pi^n) \pi^n (\mathcal{S} \otimes \text{id})\Delta \\ &= \pi^n m_{\diamond} \pi^n (\mathcal{S} \otimes \text{id})\Delta = \pi^n m_{\diamond} (\mathcal{S} \otimes \text{id})\Delta = \pi^n u\varepsilon = u\varepsilon = u\varepsilon \pi^n \end{aligned}$$

□

Note that the n -th truncated algebra (A^n, \diamond_n) is again a graded algebra. The grading is given by $(A_i^n)_i$ where $A_i^n = A_i$ for $i \leq n$ and $A_i^n = \{0\}$ for $i > n$. Hence, we get functions \exp_{\diamond_n} and \log_{\diamond_n} as described in the previous section.

3.4 Primitive and group-like elements

Definition 3.31. (Based on Section I.5. [Man06], Section 2.3 [HK14])

1. (i) The subset of *group-like elements* of a coalgebra (C, Δ) is given by

$$\text{GrLk } C := \{c \in C \setminus \{0\} \mid \Delta c = c \otimes c\}.$$

- (ii) The subset of *group-like elements* of the space of formal series C_Π of a graded coalgebra (C, Δ) is given by

$$\text{GrLk } C_\Pi := \{c \in C_\Pi \setminus \{0\} \mid \Delta c = c \otimes c\}.$$

- (iii) The subset of *n -th grade truncated group-like elements* of a coalgebra (C, Δ) graded by $(C_i)_i$ is given by

$$\text{GrLk}_n C := \{c \in C \setminus \{0\} \mid \Delta c = \pi^n(c \otimes c)\},$$

where $\pi^n : C \otimes C \rightarrow [C \otimes C]^n$ is the canonical projection.

2. (i) The subspace of *primitive elements* of a bialgebra (B, \diamond, Δ) with unit element $\mathbf{1}$ is given by

$$\text{Prim } B := \{b \in B \mid \Delta b = \mathbf{1} \otimes b + b \otimes \mathbf{1}\}.$$

We put $\text{Prim } S := S \cap \text{Prim } B$ for any subspace S of B .

- (ii) The subspace of *primitive elements* of the space of formal series B_Π of a graded bialgebra (B, \diamond, Δ) with unit element $\mathbf{1}$ is given by

$$\text{Prim } B_\Pi := \{b \in B_\Pi \mid \Delta b = \mathbf{1} \otimes b + b \otimes \mathbf{1}\}.$$

We put $\text{Prim } S := S \cap \text{Prim } B_\Pi$ for any subspace S of B_Π .

Remark 3.32.

- (i) For any $c \in \text{GrLk } C$, where (C, Δ) is a coalgebra with counit ε , we have $\varepsilon(c) = 1$ since $c \neq 0$ and

$$c = (\varepsilon \otimes \text{id})\Delta c = (\varepsilon \otimes \text{id})(c \otimes c) = \varepsilon(c)c$$

due to the counit property (2.5).

Analogously, in the case of a graded coalgebra C , we have $\varepsilon(c) = 1$ for all $c \in \text{GrLk } C_\Pi$.

- (ii) For any $b \in \text{Prim } B$, where (B, \diamond, Δ) is a coalgebra, we have $\varepsilon(b) = 0$ since

$$b = (\varepsilon \otimes \text{id})\Delta b = (\varepsilon \otimes \text{id})(\mathbf{1} \otimes b + b \otimes \mathbf{1}) = \varepsilon(\mathbf{1})b + \varepsilon(b)\mathbf{1} = b + \varepsilon(b)\mathbf{1}.$$

by the counit property (2.5) and Remark 2.10.

Analogously, in the case of a graded bialgebra B , we have $\varepsilon(b) = 0$ for all $b \in \text{Prim } B_\Pi$.

Remark 3.33. For a graded coalgebra (C, Δ) , we have

$$\pi^n \text{GrLk } C_\Pi \subseteq \text{GrLk}_n C.$$

Indeed, if $c \in \text{GrLk } C_\Pi$, then

$$\Delta \pi^n c = \pi^n \Delta c = \pi^n (c \otimes c) = \pi^n (\pi^n c \otimes \pi^n c)$$

and also $\pi^n c \neq 0$ since $\varepsilon(c) = 1$.

A *character* of an algebra (A, \diamond) is an element of $A^* \setminus \{0\}$ which is an algebra homomorphism (Section II.4. [Man06]).

Theorem 3.34. (Based on Section 2.3 [HK14]) *Let (A, \diamond) be the dual algebra of a coalgebra (C, Δ) under some duality pairing $\langle \cdot, \cdot \rangle$.*

1. *An element $c \in C$ is group-like iff $\langle \cdot, c \rangle$ is a character of (A, \diamond) .*
2. *If both (A, \diamond) and (C, Δ) are graded, then $c_1 \in C$ is group-like iff $\langle \cdot, c_1 \rangle$ is a character of (A_Π, \diamond) and $c_2 \in C_\Pi$ is group-like iff $\langle \cdot, c_2 \rangle$ is a character of (A, \diamond) .*

The proof is a basic application of the duality of products and coproducts (see Theorem 2.26) and of the considerations in Subsection 3.3.1 and is skipped here.

Theorem 3.35. (Proposition I.7.3. [Man06]) *Let $(H, \diamond, \Delta, \mathcal{S})$ be a Hopf algebra with unit map u , unit element $\mathbf{1}$ and counit ε . Then, $\mathcal{S}x = -x$ for all $x \in \text{Prim } H$. If H is furthermore connectedly graded by $(H_i)_i$, we have $\mathcal{S}x = -x$ for all $x \in \text{Prim } H_\Pi$.*

Proof. (Proof of Proposition I.7.3. [Man06]) Let $x \in \text{Prim } H$ be arbitrary. Then,

$$0 = u\varepsilon(x) = m_*(\mathcal{S} \otimes \text{id})\Delta x = m_*(\mathcal{S} \otimes \text{id})(\mathbf{1} \otimes x + x \otimes \mathbf{1}) = \mathcal{S}\mathbf{1} \diamond x + \mathcal{S}x \diamond \mathbf{1} = x + \mathcal{S}x,$$

thus $\mathcal{S}x = -x$.

The proof for $x \in \text{Prim } H_\Pi$ is completely analogous. □

Theorem 3.36. (Proposition I.7.3. [Man06]) *For a bialgebra (B, \diamond, Δ) , $\text{Prim } B$ is a sub Lie algebra of (B, \mathfrak{b}_\circ) .*

Proof. (Proof of Proposition I.7.3. [Man06]) Let $x, y \in \text{Prim } B$ be arbitrary. Then,

$$\begin{aligned} \Delta[x, y]_\circ &= \Delta(x \diamond y - y \diamond x) = \Delta x \diamond \Delta y - \Delta y \diamond \Delta x \\ &= (\mathbf{1} \otimes x + x \otimes \mathbf{1}) \diamond (\mathbf{1} \otimes y + y \otimes \mathbf{1}) - (\mathbf{1} \otimes y + y \otimes \mathbf{1}) \diamond (\mathbf{1} \otimes x + x \otimes \mathbf{1}) \\ &= \mathbf{1} \otimes (x \diamond y) + y \otimes x + x \otimes y + (x \diamond y) \otimes \mathbf{1} - \mathbf{1} \otimes (y \diamond x) - x \otimes y - y \otimes x - (y \diamond x) \otimes \mathbf{1} \\ &= \mathbf{1} \otimes (x \diamond y - y \diamond x) + (x \diamond y - y \diamond x) \otimes \mathbf{1} = \mathbf{1} \otimes [x, y]_\circ - [x, y]_\circ \otimes \mathbf{1}. \end{aligned}$$

Thus, $[x, y]_\circ \in \text{Prim } B$. □

Theorem 3.37. (Based on Theorem 1.4 and Lemma 1.5 [Reu93]) *Let $(H, \diamond, \Delta, \mathcal{S})$ be a Hopf algebra connectedly graded by $(H_i)_i$ over a field of characteristic zero, \mathfrak{R}_{H_1} the right norm bracketing operator of (H, \mathfrak{b}_\circ) restricted to $\mathbb{T}(H_1)$, D the derivation generated by*

$$Dh_n = nh_n \quad \forall h_n \in H_n,$$

furthermore $*$ the convolution product on $L(H, H)$ and $E_{H_1} : (T(H_1), \dot{\otimes}) \rightarrow (H, \diamond)$ the algebra homomorphism generated by

$$E_{H_1} h_1 = h_1 \quad \forall h_1 \in H_1.$$

Then,

$$\mathfrak{R}_{H_1} = (D * \mathcal{S})E_{H_1} \tag{3.19}$$

and

$$(D * \mathcal{S})|_{\text{Prim } H} = D|_{\text{Prim } H}. \tag{3.20}$$

Proof. (Based on the proofs of Theorem 1.4 and Lemma 1.5 [Reu93]) First of all,

$$(D * \mathcal{S})E_{H_1} \mathbf{1} = m_*(D \otimes \mathcal{S})\Delta \mathbf{1} = D\mathbf{1} \diamond \mathcal{S}\mathbf{1} = 0 = \mathfrak{R}_{H_1} \mathbf{1}$$

and, for all $v \in H_1$,

$$(D * \mathcal{S})E_{H_1} v = m_*(D \otimes \mathcal{S})\Delta v = D\mathbf{1} \otimes \mathcal{S}v + Dv \otimes \mathcal{S}\mathbf{1} = v = \mathfrak{R}_{H_1} v.$$

Assuming we have $(D * \mathcal{S})E_{H_1} \hat{u} = \mathfrak{R}_{H_1} \hat{u}$ for some $\hat{u} \in T(H_1)$ and putting $u := E_{H_1} \hat{u}$, we have for all $x \in H_1$ that

$$\begin{aligned} (D * \mathcal{S})E_{H_1}(x \dot{\otimes} \hat{u}) &= m_*(D \otimes \mathcal{S})\Delta(x \dot{\otimes} u) = \sum_{(x)} \sum_{(u)} D(x_1 \diamond u_1) \diamond \mathcal{S}(x_2 \diamond u_2) \\ &= \sum_{(x)} \sum_{(u)} (Dx_1 \diamond u_1 + x_1 \diamond Du_1) \diamond \mathcal{S}u_2 \diamond \mathcal{S}x_2 \\ &= \sum_{(u)} (D\mathbf{1} \diamond u_1 + \mathbf{1} \diamond Du_1) \diamond \mathcal{S}u_2 \diamond \mathcal{S}x + \sum_{(u)} (Dx \diamond u_1 + x \diamond Du_1) \diamond \mathcal{S}u_2 \diamond \mathcal{S}\mathbf{1} \\ &= - \sum_{(u)} Du_1 \diamond \mathcal{S}u_2 \diamond x + \sum_{(u)} x \diamond u_1 \diamond \mathcal{S}u_2 + \sum_{(u)} x \diamond Du_1 \diamond \mathcal{S}u_2 \\ &= -(D * \mathcal{S})u \diamond x + x \diamond u \varepsilon(u) + x \diamond (D * \mathcal{S})u = -\mathfrak{R}_{H_1} \hat{u} \diamond x + x \diamond \mathfrak{R}_{H_1} \hat{u} \\ &= [x, \mathfrak{R}_{H_1} \hat{u}]_{\diamond} = \mathfrak{R}_{H_1}(x \dot{\otimes} \hat{u}). \end{aligned}$$

Thus, we get (3.19) via induction. Since for all $y \in \text{Prim } H$ we have

$$(D * \mathcal{S})y = m_*(D \otimes \mathcal{S})\Delta y = D\mathbf{1} \otimes \mathcal{S}y + Dy \otimes \mathcal{S}\mathbf{1} = Dy,$$

equation (3.20) also follows. □

Corollary 3.38. (Based on Theorem 1.4 [Reu93]) *For a Hopf algebra H connectedly graded by $(H_i)_i$ over a field of characteristic zero, we have*

$$\text{Prim}\langle H_1 \rangle_{\diamond} = \langle H_1 \rangle_{\mathfrak{b}_{\diamond}}.$$

Proof. (Based on the proof of Theorem 1.4 [Reu93]) Since $\text{Prim}\langle H_1 \rangle_{\diamond}$ is a Lie algebra that contains H_1 , we have $\langle H_1 \rangle_{\mathfrak{b}_{\diamond}} \subseteq \text{Prim}\langle H_1 \rangle_{\diamond}$. On the other hand,

$$\begin{aligned} \text{Prim}\langle H_1 \rangle_{\diamond} &= D \text{Prim}\langle H_1 \rangle_{\diamond} = (D * \mathcal{S}) \text{Prim}\langle H_1 \rangle_{\diamond} \\ &\subseteq (D * \mathcal{S})\langle H_1 \rangle_{\diamond} = (D * \mathcal{S})E_{H_1} T(H_1) = \mathfrak{R}_{H_1} T(H_1) = \langle H_1 \rangle_{\mathfrak{b}_{\diamond}}. \end{aligned}$$

□

Theorem 3.39. (Based on Theorems 3.1 and 3.2 [Reu93]) *For a Hopf algebra $(H, \diamond, \Delta, \mathcal{S})$ connectedly graded by $(H_i)_i$ over a field of characteristic zero, we have*

$$\text{Prim } H_{\Pi} = \prod_{n=1}^{\infty} \text{Prim } H_n \quad \text{and} \quad \exp_{\diamond}(\text{Prim } H_{\Pi}) = \text{GrLk } H_{\Pi}.$$

Proof. (Based on the proofs of Theorems 3.1 and 3.2 [Reu93]) If x is in $\text{Prim } H_{\Pi}$, then, due to the grading,

$$\Delta \pi_n x = \pi_n \Delta x = \pi_n (\mathbf{1} \otimes x + x \otimes \mathbf{1}) = \mathbf{1} \otimes \pi_n x + \pi_n x \otimes \mathbf{1}.$$

On the other hand, of course, the spaces $\text{Prim } H_n$ are linearly independent and for $x_n \in \text{Prim } H_n$, we have

$$\Delta \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \Delta x_n = \sum_{n=1}^{\infty} (\mathbf{1} \otimes x_n + x_n \otimes \mathbf{1}) = \mathbf{1} \otimes \left(\sum_{n=1}^{\infty} x_n \right) + \left(\sum_{n=1}^{\infty} x_n \right) \otimes \mathbf{1}.$$

For any $u \in \text{Prim } H_{\Pi}$, using Theorem 3.26 2. based on

$$(\mathbf{1} \otimes u) \diamond (u \otimes \mathbf{1}) = u \otimes u = (u \otimes \mathbf{1}) \diamond (\mathbf{1} \otimes u)$$

and using Theorem 3.27 for the algebra homomorphism $\Delta : (H_{\Pi}, \diamond) \rightarrow ((H \otimes H)_{\Pi}, \underline{\diamond})$, we have that

$$\begin{aligned} \Delta \exp_{\diamond}(u) &= \exp_{\underline{\diamond}}(\Delta u) = \exp_{\underline{\diamond}}(\mathbf{1} \otimes u + u \otimes \mathbf{1}) = \exp_{\underline{\diamond}}(\mathbf{1} \otimes u) \underline{\diamond} \exp_{\underline{\diamond}}(u \otimes \mathbf{1}) \\ &= (\mathbf{1} \otimes \exp_{\diamond}(u)) \underline{\diamond} (\exp_{\diamond}(u) \otimes \mathbf{1}) = \exp_{\diamond}(u) \otimes \exp_{\diamond}(u). \end{aligned}$$

For any $v \in \text{GrLk } H_{\Pi}$, we have that $\varepsilon(v) = 1$ due to the extended counit property and thus $v \in \mathcal{G}_H$. Hence, we may compute, again using Theorems 3.26 2. and 3.27,

$$\begin{aligned} \Delta \log_{\diamond}(v) &= \log_{\underline{\diamond}}(\Delta v) = \log_{\underline{\diamond}}(v \otimes v) = \log_{\underline{\diamond}}((\mathbf{1} \otimes v) \underline{\diamond} (v \otimes \mathbf{1})) \\ &= \log_{\underline{\diamond}}(\mathbf{1} \otimes v) + \log_{\underline{\diamond}}(v \otimes \mathbf{1}) = \mathbf{1} \otimes \log_{\diamond}(v) + \log_{\diamond}(v) \otimes \mathbf{1}. \end{aligned} \tag{3.21}$$

□

Corollary 3.40. (Based on Section 2.4 [HK14]) *For a bialgebra (B, \diamond, Δ) connectedly graded by $(B_i)_i$ over a field of characteristic zero, we have*

$$\exp_{\diamond_n}(\text{Prim } B^n) = \pi^n \text{GrLk } B_{\Pi} = \text{GrLk}_n B.$$

Proof. For all $v \in \text{GrLk}_n H$, we have, by Theorem 3.27 and the same calculation as in (3.21), that

$$\begin{aligned} \Delta \log_{\diamond_n}(v) &= \log_{\underline{\diamond}_n}(\Delta v) = \log_{\underline{\diamond}_n}(\pi^n (v \otimes v)) = \pi^n \log_{\underline{\diamond}}(v \otimes v) = \pi^n (\mathbf{1} \otimes \log_{\diamond}(v) + \log_{\diamond}(v) \otimes \mathbf{1}) \\ &= \mathbf{1} \otimes \pi^n \log_{\diamond}(v) + \pi^n \log_{\diamond}(v) \otimes \mathbf{1} = \mathbf{1} \otimes \log_{\diamond_n}(v) + \log_{\diamond_n}(v) \otimes \mathbf{1}. \end{aligned}$$

Thus, $\log_{\diamond_n}(\text{GrLk}_n H) \subseteq \text{Prim } B^n$, implying $\text{GrLk}_n H \subseteq \exp_{\diamond_n}(\text{Prim } B^n)$. Again applying Theorem 3.27, this time for the exponential map, we compute

$$\exp_{\diamond_n}(\text{Prim } B^n) = \exp_{\diamond_n}(\pi^n \text{Prim } B_{\Pi}) = \pi^n \exp_{\diamond}(\text{Prim } B_{\Pi}) = \pi^n \text{GrLk } B_{\Pi}.$$

Altogether, using Remark 3.33, we finally get

$$\text{GrLk}_n H \subseteq \exp_{\diamond_n}(\text{Prim } B^n) = \pi^n \text{GrLk } B_{\Pi} \subseteq \text{GrLk}_n H.$$

□

Example 3.41.

1. In the case of polynomials, due to Corollary 3.38,

$$\text{Prim } \mathcal{P}_d = \text{Prim}\langle \mathcal{P}_{d,1} \rangle_{\bullet} = \langle \mathcal{P}_{d,1} \rangle_{\mathfrak{b}_{\bullet}} = \mathcal{P}_{d,1}$$

since \bullet is commutative and hence also

$$\text{Prim}(\mathcal{P}_d)_{\Pi} = \mathcal{P}_{d,1} = \text{span}\{X_i | 1 \leq i \leq d\}.$$

Theorem 3.39 then implies $\text{GrLk}(\mathcal{P}_d)_{\Pi} = \{\exp_{\bullet}(\sum_{i=1}^d a_i X_i) | (a_i)_i \in \mathbb{R}\}$.

2. For the concatenation algebra, we again have $\langle V \rangle_{\dot{\otimes}} = \text{T}(V)$, hence

$$\text{Prim}(\text{T}(V), \Delta_{\sqcup}) = \langle V \rangle_{\mathfrak{b}_{\dot{\otimes}}}.$$

The elements of $\text{Prim}(\text{T}(V), \Delta_{\sqcup})$ are called *Lie polynomials* over V (Section 1.3 [Reu93]). We put $\text{T}((V)) := \text{T}(V)_{\Pi}$ and call

$$\text{Prim}(\text{T}((V)), \Delta_{\sqcup}) = \prod_{i=1}^{\infty} (\langle V \rangle_{\mathfrak{b}_{\dot{\otimes}}} \cap V^{\dot{\otimes} i})$$

the space of over V (Section 3.1 [Reu93]¹).

3. For the shuffle algebra, $\langle V \rangle_{\sqcup} = \text{T}(V)$ is a proper subspace of $\text{T}(V)$. Therefore, we may only conclude from Corollary 3.38 that

$$\text{Prim}(\langle V \rangle_{\sqcup}, \Delta_{\dot{\otimes}}) = \langle V \rangle_{\mathfrak{b}_{\sqcup}} = V,$$

which follows from \sqcup being commutative. But, looking at (3.15), we see that every summand in the expansion of $\Delta_{\dot{\otimes}} u$ for some element of the form $u = v_1 \dot{\otimes} \cdots \dot{\otimes} v_n$, $v_i \in V$ is specific to u , hence none of those summands can be canceled out by $\Delta_{\dot{\otimes}} w$ for an element w of the same form as u , but linearly independent. Hence, there cannot be primitive elements of $\Delta_{\dot{\otimes}}$ in $V^{\dot{\otimes} n}$ for $n > 1$. Therefore,

$$\text{Prim}(\text{T}((V)), \Delta_{\dot{\otimes}}) = \text{Prim}(\text{T}(V), \Delta_{\dot{\otimes}}) = V$$

and, by Theorem 3.39, we get $\text{GrLk}(\text{T}((V)), \Delta_{\dot{\otimes}}) = \exp_{\sqcup}(V)$.

Interestingly, the example of $(\text{T}((K^2)), \dot{\otimes}, \Delta_{\sqcup})$, K being a field of characteristic zero, can be used to show the following.

Theorem 3.42. (Based on Corollaries 3.3 and 3.4 [Reu93]) *For an algebra (A, \diamond) connectedly graded by $(A_i)_i$ over a field of characteristic zero, $\exp_{\diamond}(W)$ is a subgroup of $(\mathcal{G}_A, \diamond)$ for any sub Lie algebra $W = \prod_{i=0}^{\infty} (W \cap A_i)$ of $(\mathfrak{g}_A, \mathfrak{b}_{\diamond})$.*

Proof. (Based on the proof of Corollary 3.3 [Reu93]) Let $\{e_1, e_2\}$ be a basis of K^2 . Then, we of course have $e_1, e_2 \in \text{Prim}(\text{T}((K^2)), \Delta_{\sqcup})$ and hence $\exp(e_1), \exp(e_2) \in \text{GrLk}(\text{T}((K^2)), \Delta_{\sqcup})$. We compute

$$\begin{aligned} \Delta_{\sqcup}(\exp_{\dot{\otimes}}(e_1) \dot{\otimes} \exp_{\dot{\otimes}}(e_2)) &= \Delta_{\sqcup} \exp_{\dot{\otimes}}(e_1) \dot{\otimes} \Delta_{\sqcup} \exp_{\dot{\otimes}}(e_2) \\ &= (\exp_{\dot{\otimes}}(e_1) \otimes \exp_{\dot{\otimes}}(e_1)) \dot{\otimes} (\exp_{\dot{\otimes}}(e_2) \otimes \exp_{\dot{\otimes}}(e_2)) \\ &= (\exp_{\dot{\otimes}}(e_1) \dot{\otimes} \exp_{\dot{\otimes}}(e_2)) \otimes (\exp_{\dot{\otimes}}(e_1) \dot{\otimes} \exp_{\dot{\otimes}}(e_2)). \end{aligned}$$

¹Note that in the case where V is infinite dimensional, the space of Lie series defined in [Reu93] is strictly larger than the one presented here. According to our definition, Lie series always reduce to finite sums when projected down to $\text{T}^n(V)$, which is not the case in [Reu93]. The definitions agree if V is finite dimensional.

Thus,

$$\exp_{\dot{\otimes}}(e_1) \dot{\otimes} \exp_{\dot{\otimes}}(e_2) \in \text{GrLk}(\mathbb{T}((K^2)), \Delta_{\mathfrak{u}}) = \exp_{\dot{\otimes}}(\text{Prim } \mathbb{T}((K^2))) = \exp_{\dot{\otimes}}(\langle (e_1, e_2) \rangle_{\mathfrak{b}_{\dot{\otimes}}})_{\Pi}$$

Now, let $v_1, v_2 \in \exp_{\diamond}(W)$ be arbitrary. Let $\Lambda : (\mathbb{T}((K^2)), \dot{\otimes}) \rightarrow (A_{\Pi}, \diamond)$ be the unique algebra homomorphism such that $\Lambda e_1 = \log_{\diamond} v_1$, $\Lambda e_2 = \log_{\diamond} v_2$ and $\pi_n \Lambda x = \Lambda \pi_n x$ for all $x \in \mathbb{T}((K^2))$. Then,

$$\Lambda \exp_{\dot{\otimes}}(e_i) = \Lambda \sum_{n=0}^{\infty} \frac{e_i^{\dot{\otimes} n}}{n!} = \sum_{n=0}^{\infty} \frac{\Lambda(e_i^{\dot{\otimes} n})}{n!} = \sum_{n=0}^{\infty} \frac{(\Lambda e_i)^{\diamond n}}{n!} = \exp_{\diamond}(\Lambda e_i) = v_i,$$

therefore

$$v_1 \diamond v_2 = (\Lambda \exp_{\dot{\otimes}}(e_1)) \diamond (\Lambda \exp_{\dot{\otimes}}(e_2)) = \Lambda(\exp_{\dot{\otimes}}(e_1) \dot{\otimes} \exp_{\dot{\otimes}}(e_2))$$

and finally

$$\begin{aligned} v_1 \diamond v_2 &\in \Lambda \exp_{\dot{\otimes}}(\langle (e_1, e_2) \rangle_{\mathfrak{b}_{\dot{\otimes}}})_{\Pi} = \exp_{\diamond}(\Lambda \langle (e_1, e_2) \rangle_{\mathfrak{b}_{\dot{\otimes}}})_{\Pi} = \exp_{\diamond}(\langle \Lambda e_1, \Lambda e_2 \rangle_{\mathfrak{b}_{\dot{\otimes}}})_{\Pi} \\ &= \exp_{\diamond}(\langle \log_{\diamond} v_1, \log_{\diamond} v_2 \rangle_{\mathfrak{b}_{\diamond}})_{\Pi} \subseteq \exp_{\diamond}(W). \end{aligned}$$

Since for any $v \in \exp_{\diamond}(W)$, we have $v^{\diamond-1} = \exp_{\diamond}(-\log_{\diamond}(v)) \in \exp_{\diamond}(W)$, the proof of $\exp_{\diamond}(W)$ being a subgroup is complete. \square

Remark 3.43. The Lie series $\log_{\dot{\otimes}}(\exp_{\dot{\otimes}}(a) \dot{\otimes} \exp_{\dot{\otimes}}(b))$ (Corollary 3.4 [Reu93]) is called the *Baker-Campbell-Hausdorff series* of $a, b \in \text{Prim}(\mathbb{T}((V)), \Delta_{\mathfrak{u}})$. See Chapter 3 [Reu93] for more on this series.

Remark 3.44. (Based on Corollary 3.3 [Reu93]) We now especially know that for every Hopf algebra $(H, \diamond, \Delta, \mathcal{S})$ connectedly graded by $(H_i)_i$, the set of group-like elements $\text{GrLk } H_{\Pi} = \exp_{\diamond} \text{Prim } H_{\Pi}$ is a subgroup of $(\mathcal{G}_H, \diamond)$. Since

$$x \diamond \mathcal{S}x = m_*(\text{id} \otimes \mathcal{S})(x \otimes x) = m_*(\text{id} \otimes \mathcal{S})\Delta x = u\varepsilon(x) = u(1) = \mathbf{1},$$

where we made use of the antipode property (2.14) and of Remark 3.32, we have $x^{\diamond-1} = \mathcal{S}x$ for all $x \in \text{GrLk } H_{\Pi}$.

3.5 Connes-Kreimer Hopf algebra of trees and forests

3.5.1 Trees and forests

In the following definition, we formally introduce the set of *forests* \mathcal{F}_I and the subset of *trees* \mathcal{T}_I labeled or decorated by indices from some set I .

Definition 3.45. (Based on Section 2.2 [HK14]) For a non-empty set I let $\mathcal{T}_I \subset \mathcal{F}_I$ be sets, $\mathbf{1} \in \mathcal{F}_I$ an element, $\odot : \mathcal{F}_I \times \mathcal{F}_I \rightarrow \mathcal{F}_I$ an associative and commutative binary operation, $|\cdot| : \mathcal{F}_I \rightarrow \mathbb{N}_0$ a function and $[\cdot]_i : \mathcal{F}_I \rightarrow \mathcal{T}_I, i \in I$ injective functions such that the following conditions are fulfilled.

- (i) $\mathbf{1} \odot \zeta = \zeta \odot \mathbf{1} = \zeta \quad \forall \zeta \in \mathcal{F}_I,$
- (ii) \mathcal{T}_I is the disjoint union of the sets $[\mathcal{F}_I]_i, i \in I,$
- (iii) For every $\zeta \in \mathcal{F}_I \setminus \{\mathbf{1}\},$ there is an $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in \mathcal{T}_I$ such that $\zeta = \tau_1 \odot \dots \odot \tau_n$ and this factorization is unique up to commutativity,
- (iv) $|\zeta_1 \odot \zeta_2| = |\zeta_1| + |\zeta_2| \quad \forall \zeta_1, \zeta_2 \in \mathcal{F}_I,$
- (v) $||\zeta]_i| = |\zeta| + 1 \quad \forall \zeta \in \mathcal{F}_I, i \in I,$
- (vi) $|\zeta| = 0 \implies \zeta = \mathbf{1}.$

We call $\mathbf{1}$ the *empty forest*. If I consists only of one element $j,$ we put $\mathcal{F} := \mathcal{F}_{\{j\}}, \mathcal{T} := \mathcal{T}_{\{j\}}$ and $[\cdot] := [\cdot]_j$ and call \mathcal{F} the set of *undecorated forests* and \mathcal{T} the set of *undecorated trees*.

3.5.2 Commutative Hopf algebra

Let $\mathcal{F}_{I,n} := \{\zeta \in \mathcal{F}_I \mid |\zeta| = n\}$ and $\mathcal{T}_{I,n} := \mathcal{T}_I \cap \mathcal{F}_{I,n}.$ We extend \odot bilinearly from $\mathcal{F}_I \times \mathcal{F}_I$ to $\langle \mathcal{F}_I \rangle \times \langle \mathcal{F}_I \rangle$ and all $[\cdot]_i$ linearly from \mathcal{F}_I to $\langle \mathcal{F}_I \rangle.$

Lemma 3.46. $(\langle \mathcal{F}_I \rangle, \odot)$ is a commutative unitary algebra connectedly graded by $(\langle \mathcal{F}_{I,n} \rangle)_n.$

Definition 3.47. (Equations (48) to (50) [CK98], Equation (2.3) [HK14]) Let $\Delta_* : \langle \mathcal{F}_I \rangle \rightarrow \langle \mathcal{F}_I \rangle \otimes \langle \mathcal{F}_I \rangle$ be the linear map recursively generated by

$$\Delta_* \mathbf{1} := \mathbf{1} \otimes \mathbf{1}, \quad \Delta_* [\zeta]_i := [\zeta]_i \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i) \Delta_* \zeta, \quad \Delta_* (\zeta_1 \odot \zeta_2) := \Delta_* \zeta_1 \odot \Delta_* \zeta_2, \quad (3.22)$$

n	\mathcal{F}_n	\rightarrow	\mathcal{T}_{n+1}
0	$\mathbf{1}$	\mapsto	\bullet
1	\bullet	\mapsto	\vdots
2	$\bullet\bullet$ \vdots	\mapsto	\vee \vdots
3	\dots $\bullet\bullet$ \vee \vdots	\mapsto	$\vee\vee$ \vee \vee \vdots

Table 3.1: The action of $[\cdot].$

where $\zeta, \zeta_1, \zeta_2 \in \mathcal{F}_I$ and \odot is again the canonical associative product on $\langle \mathcal{F}_I \rangle \otimes \langle \mathcal{F}_I \rangle$ (cf. (2.9)) generated by

$$(\zeta_1 \otimes \eta_1) \odot (\zeta_2 \otimes \eta_2) := (\zeta_1 \odot \zeta_2) \otimes (\eta_1 \odot \eta_2).$$

Furthermore, let $\varepsilon \in \langle \mathcal{F}_I \rangle^*$ be generated by

$$\varepsilon(\zeta) := \delta_{1, \zeta}, \quad \zeta \in \mathcal{F}_I.$$

Example 3.48. In order to find the value of $\Delta_* \mathfrak{I}_1^3 \mathfrak{I}_3^3$, we first compute generally those for the first two linear trees

$$\begin{aligned} \Delta_* \bullet^i &= \bullet^i \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i) \Delta_* \mathbf{1} = \bullet^i \otimes \mathbf{1} + \mathbf{1} \otimes \bullet^i, \\ \Delta_* \mathfrak{I}_j^i &= \mathfrak{I}_j^i \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_j) \Delta_* \bullet^i = \mathfrak{I}_j^i \otimes \mathbf{1} + \bullet^i \otimes \bullet^j + \mathbf{1} \otimes \mathfrak{I}_j^i, \end{aligned}$$

and then specifically

$$\begin{aligned} \Delta_* \mathfrak{I}_1^3 \mathfrak{I}_3^3 &= \Delta_* \mathfrak{I}_2^3 \odot \Delta_* \mathfrak{I}_3^3 \\ &= \mathfrak{I}_2^3 \otimes \mathbf{1} + \bullet_3 \mathfrak{I}_2^3 \otimes \bullet^2 + \mathfrak{I}_2^3 \otimes \mathfrak{I}_3^3 + \bullet_2 \mathfrak{I}_2^3 \otimes \bullet^3 + \bullet^2 \bullet^3 \otimes \bullet^2 \bullet^3 + \bullet^2 \otimes \bullet_3 \mathfrak{I}_2^3 + \mathfrak{I}_2^3 \otimes \mathfrak{I}_3^3 \\ &\quad + \bullet^3 \otimes \bullet_2 \mathfrak{I}_2^3 + \mathbf{1} \otimes \mathfrak{I}_2^3 \mathfrak{I}_3^3, \\ \Delta_* \mathfrak{I}_1^3 \mathfrak{I}_3^3 &= \mathfrak{I}_1^3 \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_1) \Delta_* \mathfrak{I}_2^3 \mathfrak{I}_3^3 \\ &= \mathfrak{I}_1^3 \otimes \mathbf{1} + \mathfrak{I}_2^3 \otimes \bullet^1 + \bullet_3 \mathfrak{I}_2^3 \otimes \mathfrak{I}_1^2 + \mathfrak{I}_2^3 \otimes \mathfrak{I}_1^3 + \bullet_2 \mathfrak{I}_2^3 \otimes \mathfrak{I}_1^3 + \bullet^2 \bullet^3 \otimes \mathfrak{I}_1^3 + \bullet^2 \otimes \bullet_3 \mathfrak{I}_1^3 + \mathfrak{I}_2^3 \otimes \mathfrak{I}_1^3 \\ &\quad + \bullet^3 \otimes \bullet_2 \mathfrak{I}_1^3 + \mathbf{1} \otimes \mathfrak{I}_1^3 \mathfrak{I}_2^3. \end{aligned}$$

By identification of the labels 2 and 3 we furthermore get

$$\Delta_* \mathfrak{I}_1^3 \mathfrak{I}_3^3 = \mathfrak{I}_1^3 \otimes \mathbf{1} + \mathfrak{I}_2^3 \otimes \bullet^1 + 2 \bullet_2 \mathfrak{I}_2^3 \otimes \mathfrak{I}_1^2 + 2 \mathfrak{I}_2^3 \otimes \mathfrak{I}_1^3 + \bullet^2 \bullet^2 \otimes \mathfrak{I}_1^2 + 2 \bullet^2 \otimes \bullet_3 \mathfrak{I}_1^2 + \mathbf{1} \otimes \mathfrak{I}_1^3 \mathfrak{I}_2^3,$$

and identifying all the labels we have the undecorated case

$$\Delta_* \mathfrak{I}^3 = \mathfrak{I}^3 \otimes \mathbf{1} + \mathfrak{I}^2 \otimes \bullet + 2 \bullet \otimes \mathfrak{I}^2 + 2 \mathfrak{I}^2 \otimes \mathfrak{I} + \bullet \bullet \otimes \mathfrak{I} + 2 \bullet \otimes \mathfrak{I}^2 + \mathbf{1} \otimes \mathfrak{I}^3.$$

Theorem 3.49. (Based on Equations (52) and (53) [CK98], Section 5.1 [Foi02], Section 2.2 [HK14]) $(\langle \mathcal{F}_I \rangle, \odot, \Delta_*, \mathcal{S}_\circ^*)$ is a commutative Hopf algebra connectedly graded by $(\langle \mathcal{F}_{I,n} \rangle)_n$ with counit ε , where the linear map $\mathcal{S}_\circ^* : \langle \mathcal{F}_I \rangle \rightarrow \langle \mathcal{F}_I \rangle$ is recursively generated by

$$\mathcal{S}_\circ^* \mathbf{1} := \mathbf{1}, \quad \mathcal{S}_\circ^*(\zeta_1 \odot \zeta_2) := \mathcal{S}_\circ^* \zeta_1 \odot \mathcal{S}_\circ^* \zeta_2, \quad \mathcal{S}_\circ^*[\zeta]_i := -m_\circ(\mathcal{S}_\circ^* \otimes [\cdot]_i) \Delta_* \zeta. \quad (3.23)$$

Proof.

1. $(\langle \mathcal{F}_I \rangle, \Delta_*)$ is a coalgebra. (Section 5.1 [Foi02]) Obviously, $(\Delta_* \otimes \text{id}) \Delta_* \mathbf{1} = (\text{id} \otimes \Delta_*) \Delta_* \mathbf{1}$. If $\zeta_1, \zeta_2 \in \mathcal{F}_I$ are such that $(\Delta_* \otimes \text{id}) \Delta_* \zeta_i = (\text{id} \otimes \Delta_*) \Delta_* \zeta$, Lemma 2.11 implies that

$$(\Delta_* \otimes \text{id}) \Delta_* (\zeta_1 \odot \zeta_2) = (\text{id} \otimes \Delta_*) \Delta_* (\zeta_1 \odot \zeta_2)$$

since Δ_* is an algebra homomorphism by definition (see (3.22)). Finally, if $(\Delta_* \otimes \text{id}) \Delta_* \zeta = (\text{id} \otimes \Delta_*) \Delta_* \zeta$, we also have

$$\begin{aligned} (\Delta_* \otimes \text{id}) \Delta_* [\zeta]_i &= [\zeta]_i \otimes \mathbf{1} \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i \otimes \text{id}) \Delta_* \zeta + (\text{id} \otimes \text{id} \otimes [\cdot]_i) (\Delta_* \otimes \text{id}) \Delta_* \zeta \\ &= [\zeta]_i \otimes \mathbf{1} \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i \otimes \text{id}) \Delta_* \zeta + (\text{id} \otimes \text{id} \otimes [\cdot]_i) (\text{id} \otimes \Delta_*) \Delta_* \zeta \\ &= [\zeta]_i \otimes \mathbf{1} \otimes \mathbf{1} + (\text{id} \otimes \Delta_*) (\text{id} \otimes [\cdot]_i) \Delta_* \zeta = (\text{id} \otimes \Delta_*) \Delta_* [\zeta]_i. \end{aligned}$$

Hence, we get via induction that $(\Delta_* \otimes \text{id}) \Delta_* \zeta = (\text{id} \otimes \Delta_*) \Delta_* \zeta$ for all $\zeta \in \mathcal{F}_I$ and by linearity we conclude that Δ_* is coassociative.

2. ε is a counit. First of all, we have $(\varepsilon \otimes \text{id})\Delta_*\mathbf{1} = \varepsilon(\mathbf{1})\mathbf{1} = \mathbf{1} = (\text{id} \otimes \varepsilon)\Delta_*\mathbf{1}$. If we assume that $(\varepsilon \otimes \text{id})\Delta_*\zeta = \zeta = (\text{id} \otimes \varepsilon)\Delta_*\zeta$ holds for some $\zeta \in \langle \mathcal{F}_I \rangle$, we get

$$(\varepsilon \otimes \text{id})\Delta_*[\zeta]_i = \varepsilon([\zeta]_i)\mathbf{1} + [\cdot]_i(\varepsilon \otimes \text{id})\Delta_*\zeta = [\zeta]_i$$

and

$$(\text{id} \otimes \varepsilon)\Delta_*[\zeta]_i = \varepsilon(\mathbf{1})[\zeta]_i + (\text{id} \otimes \varepsilon[\cdot]_i)\Delta_*\zeta = [\zeta]_i,$$

since $\varepsilon[\cdot]_i = 0$ by definition. Finally, if we assume that for some $\zeta_1, \zeta_2 \in \langle \mathcal{F}_I \rangle$ we have $(\varepsilon \otimes \text{id})\Delta_*\zeta_j = \zeta_j = (\text{id} \otimes \varepsilon)\Delta_*\zeta_j$, we get

$$(\varepsilon \otimes \text{id})\Delta_*(\zeta_1 \odot \zeta_2) = \zeta_1 \odot \zeta_2 = (\text{id} \otimes \varepsilon)\Delta_*(\zeta_1 \odot \zeta_2)$$

by again applying Lemma 2.11, since ε is an algebra homomorphism. Indeed,

$$\varepsilon(\zeta_1)\varepsilon(\zeta_2) = \delta_{\mathbf{1}, \zeta_1} \delta_{\mathbf{1}, \zeta_2} = \delta_{\mathbf{1}, \zeta_1 \odot \zeta_2} = \varepsilon(\zeta_1 \odot \zeta_2).$$

Hence, by induction the equation

$$(\varepsilon \otimes \text{id})\Delta_*\zeta = \zeta = (\text{id} \otimes \varepsilon)\Delta_*\zeta$$

holds for all $\zeta \in \langle \mathcal{F}_I \rangle$. The counit property (2.5) then follows via linearity.

3. $(\langle \mathcal{F}_I \rangle, \odot, \Delta_*)$ is a bialgebra. We have $\Delta_*(\zeta_1 \odot \zeta_2) = \Delta_*\zeta_1 \odot \Delta_*\zeta_2$ for all $\zeta_1, \zeta_2 \in \langle \mathcal{F}_I \rangle$ and $\Delta_*\mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ by definition of Δ_* . Also, ε is an algebra homomorphism as we already saw. Thus, $(\langle \mathcal{F}_I \rangle, \odot, \Delta_*)$ is a bialgebra.
4. $(\langle \mathcal{F}_I \rangle, \odot, \Delta_*, \mathcal{S}_\odot^*)$ is a connected graded Hopf algebra. The fact that $(\langle \mathcal{F}_{I,n} \rangle)_n$ is a grading for the bialgebra $(\langle \mathcal{F}_I \rangle, \odot, \Delta_*)$ is again easily shown by an induction over the recursive definition of Δ_* . It is also connected since $\langle \mathcal{F}_{I,0} \rangle = \langle \mathbf{1} \rangle$. By Theorem 2.20 we thus have that $(\langle \mathcal{F}_I \rangle, \odot, \Delta_*, \mathcal{S})$ is a connected graded Hopf algebra with some unique antipode \mathcal{S} . Using $\varepsilon[\cdot]_i = 0$, the antipode property (2.14) and (3.22), we get

$$\mathcal{S}[\zeta]_i = -u\varepsilon([\zeta]_i) + \mathcal{S}[\zeta]_i = -m_\odot(\mathcal{S} \otimes \text{id})\Delta_*[\zeta]_i + \mathcal{S}[\zeta]_i = -m_\odot(\mathcal{S} \otimes [\cdot]_i)\Delta_*\zeta.$$

Since the antipode is an algebra antimorphism by Theorem 2.16 and \odot is commutative, we also have that

$$\mathcal{S}(\zeta_1 \odot \zeta_2) = \mathcal{S}\zeta_1 \odot \mathcal{S}\zeta_2$$

and $\mathcal{S}\mathbf{1} = \mathbf{1}$. Thus, \mathcal{S} is given by the recursion (3.23), i.e. $\mathcal{S} = \mathcal{S}_\odot^*$.

□

Remark 3.50. Due to Theorem 2.20, we may instead of the second expression in (3.23) use

$$\mathcal{S}_\odot^*[\zeta]_i := -[\zeta]_i - m_\odot(\text{id} \otimes \mathcal{S}_\odot^*)\tilde{\Delta}_*[\zeta]_i,$$

which has the advantage that here, we do not need to calculate the antipode values for forests with more than one tree in order to get the antipode values for trees.

Theorem 3.51. (Based on Equation (51) [CK98], Equation (4) [Foi02], Section II.9.3. [Man06], Remark 2.9. [HK14]) Δ_* admits the representation

$$\Delta_*\zeta = \sum_{(C,T) \in \text{Cuts}(\zeta)} c_\zeta(C,T) C \otimes T, \quad (3.24)$$

where $(\text{Cuts}(\zeta))_{\zeta \in \mathcal{F}_I}$ is a family of finite subsets of $\mathcal{F}_I \times \mathcal{F}_I$ recursively defined by

$$\begin{aligned} \text{Cuts}(\mathbf{1}) &:= \{(\mathbf{1}, \mathbf{1})\}, & \text{Cuts}([\zeta]_i) &:= \{(C, [T]_i) \mid (C, T) \in \text{Cuts}(\zeta)\} \cup \{([\zeta]_i, \mathbf{1})\}, \\ \text{Cuts}(\zeta_1 \zeta_2) &:= \{(C_1 C_2, T_1 T_2) \mid (C_1, T_1) \in \text{Cuts}(\zeta_1), (C_2, T_2) \in \text{Cuts}(\zeta_2)\} \end{aligned}$$

and $(c_\zeta(\cdot, \cdot))_{\zeta \in \mathcal{F}_I}$ is a family of functions $(\text{Cuts}(\zeta) \rightarrow \mathbb{N})_{\zeta \in \mathcal{F}_I}$ recursively defined by

$$\begin{aligned} c_\tau(\tau, \mathbf{1}) &:= 1, & c_{[\zeta]_i}(C, [T]_i) &:= c_\zeta(C, T), \\ c_{\zeta_1 \zeta_2}(C, T) &:= \sum_{\substack{(C_1, T_1) \in \text{Cuts}(\zeta_1), (C_2, T_2) \in \text{Cuts}(\zeta_2): \\ C_1 C_2 = C, T_1 T_2 = T}} c_{\zeta_1}(C_1, T_1) c_{\zeta_2}(C_2, T_2). \end{aligned}$$

Proof. First of all, we have

$$\Delta_* \mathbf{1} = \mathbf{1} \otimes \mathbf{1} = c_{\mathbf{1}}(\mathbf{1}, \mathbf{1}) \mathbf{1} \otimes \mathbf{1} = \sum_{(C, T) \in \text{Cuts}(\mathbf{1})} c_\zeta(C, T) C \otimes T.$$

Assuming $\Delta_* \zeta$ admits the representation (3.24) for some $\zeta \in \mathcal{F}_I$, we get

$$\begin{aligned} \Delta_* [\zeta]_i &= [\zeta]_i \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i) \Delta_* \zeta = c_{[\zeta]_i}([\zeta]_i, \mathbf{1}) [\zeta]_i \otimes \mathbf{1} + \sum_{(C, T) \in \text{Cuts}(\zeta)} c_\zeta(C, T) C \otimes [T]_i \\ &= c_{[\zeta]_i}([\zeta]_i, \mathbf{1}) [\zeta]_i \otimes \mathbf{1} + \sum_{(C, T) \in \text{Cuts}(\zeta)} c_{[\zeta]_i}(C, [T]_i) C \otimes [T]_i \\ &= \sum_{(\bar{C}, \bar{T}) \in \text{Cuts}([\zeta]_i)} c_{[\zeta]_i}(\bar{C}, \bar{T}) \bar{C} \otimes \bar{T}. \end{aligned}$$

Assuming $\Delta_* \zeta_1$ and $\Delta_* \zeta_2$ admit the representation (3.24) for some $\zeta_1, \zeta_2 \in \mathcal{F}_I$, we get

$$\begin{aligned} \Delta_*(\zeta_1 \zeta_2) &= \Delta_* \zeta_1 \odot \Delta_* \zeta_2 = \sum_{(C_1, T_1) \in \text{Cuts}(\zeta_1)} \sum_{(C_2, T_2) \in \text{Cuts}(\zeta_2)} c_{\zeta_1}(C_1, T_1) c_{\zeta_2}(C_2, T_2) C_1 C_2 \otimes T_1 T_2 \\ &= \sum_{(C_1, T_1) \in \text{Cuts}(\zeta_1)} \sum_{(C_2, T_2) \in \text{Cuts}(\zeta_2)} c_{\zeta_1 \zeta_2}(C_1 C_2, T_1 T_2) C_1 C_2 \otimes T_1 T_2 \\ &= \sum_{(C, T) \in \text{Cuts}(\zeta_1 \zeta_2)} c_{\zeta_1 \zeta_2}(C, T) C \otimes T. \end{aligned}$$

The claim then follows inductively. □

Referring to Section 2. [CK98], Section 4.1 [Foi02] and Section II.9.3. [Man06], we may interpret the elements of $\text{Cuts}(\zeta)$ as the set of all cuts (C, T) of the forest ζ into an upper part, the *crown* C , and a lower part, the *trunk* T . Such cuts are called *admissible cuts*. The coefficient $c_\zeta(C, T)$ then gives the number of possibilities to draw the cut (C, T) into a fixed planar drawing of the forest ζ . The special case $(\mathbf{1}, \zeta) \in \text{Cuts}(\zeta)$ is called the *empty cut* and $(\zeta, \mathbf{1}) \in \text{Cuts}(\zeta)$ is called the *full cut*.

Theorem 3.52. S_\circ^* admits the representation

$$S_\circ^* \zeta = \sum_{S \in \text{Splits}(\zeta)} s_\zeta(S) S, \quad (3.25)$$

where $(\text{Splits}(\zeta))_{\zeta \in \mathcal{F}_I}$ is a family of finite subsets of \mathcal{F}_I recursively defined by

$$\text{Splits}(\mathbf{1}) := \{\mathbf{1}\}, \quad \text{Splits}(\zeta) := \{CS \mid \exists T \neq \zeta : S \in \text{Splits}(T), (C, T) \in \text{Cuts}(\zeta)\}$$

and $(s_\zeta(\cdot))_{\zeta \in \mathcal{F}_I}$ is a family of functions $(\text{Splits}(\zeta) \rightarrow \mathbb{Z})_{\zeta \in \mathcal{F}_I}$ recursively defined by

$$s_{\mathbf{1}}(\mathbf{1}) := 1, \quad s_{[\zeta]_i}(\bar{S}) := - \sum_{\substack{(C, T) \in \text{Cuts}([\zeta]_i), S \in \text{Splits}(T): \\ T \neq [\zeta]_i, CS = \bar{S}}} s_T(S) c_{[\zeta]_i}(C, T),$$

$$s_{\zeta_1 \zeta_2}(S) := \sum_{\substack{S_1 \in \text{Splits}(\zeta_1), S_2 \in \text{Splits}(\zeta_2): \\ S_1 S_2 = S}} s_{\zeta_1}(S_1) s_{\zeta_2}(S_2).$$

Proof.

1. $\text{Splits}(\zeta_1 \zeta_2) = \text{Splits}(\zeta_1) \odot \text{Splits}(\zeta_2)$ for all $\zeta_1, \zeta_2 \in \mathcal{F}_I$. We first have

$$\text{Splits}(\mathbf{1}) \odot \text{Splits}(\zeta) = \{\mathbf{1}\} \odot \text{Splits}(\zeta) = \text{Splits}(\zeta) = \text{Splits}(\mathbf{1}\zeta)$$

for all $\zeta \in \mathcal{F}_I$, in particular for all trees ζ .

Assuming the claim holds for all $\zeta_1, \zeta_2 \in \mathcal{F}_I$ with $|\zeta_1| + |\zeta_2| \leq n$, we get for all $\eta_1, \eta_2 \in \mathcal{F}_I$ with $|\eta_1| + |\eta_2| \leq n + 1$ that

$$\begin{aligned} \text{Splits}(\eta_1 \eta_2) &= \{CS \mid \exists T \neq \eta_1 \eta_2 : S \in \text{Splits}(T), (C, T) \in \text{Cuts}(\eta_1 \eta_2)\} \\ &= \{C_1 C_2 S \mid \exists T_1 T_2 \neq \eta_1 \eta_2 : S \in \text{Splits}(T_1 T_2), (C_1, T_1) \in \text{Cuts}(\eta_1), \\ &\quad (C_2, T_2) \in \text{Cuts}(\eta_2)\} \\ &= \{C_1 C_2 S_1 S_2 \mid \exists T_1 T_2 \neq \eta_1 \eta_2 : S_1 \in \text{Splits}(T_1), S_2 \in \text{Splits}(T_2), \\ &\quad (C_1, T_1) \in \text{Cuts}(\eta_1), (C_2, T_2) \in \text{Cuts}(\eta_2)\} \\ &= \{C_1 C_2 S_1 S_2 \mid \exists T_1 \neq \eta_1, T_2 \neq \eta_2 : S_1 \in \text{Splits}(T_1), S_2 \in \text{Splits}(T_2), \\ &\quad (C_1, T_1) \in \text{Cuts}(\eta_1), (C_2, T_2) \in \text{Cuts}(\eta_2)\} \\ &= \text{Splits}(\eta_1) \odot \text{Splits}(\eta_2), \end{aligned}$$

where in the third equality we used the inductive assumption with the fact that for $(C, T) \in \text{Cuts}(\zeta)$, we have $|T| \geq |\zeta|$ if and only if $T = \zeta$. In order to see the \subseteq part of the fourth equality, the case of either $T_1 = \eta_1$ or $T_2 = \eta_2$ needs further explanation. Without loss of generality, assume $T_1 = \eta_1$, then we have $C_1 = \mathbf{1}$ and since $S_1 \in \text{Splits}(T_1) = \text{Splits}(\eta_1)$, there is $(C'_1, T'_1) \in \text{Cuts}(\eta_1)$ and $S'_1 \in \text{Splits}(T'_1)$ such that $T'_1 \neq \eta_1$ and $C'_1 S'_1 = S_1 = C_1 S_1$. Hence, this case is indeed included in what comes after the fourth equality.

The claim then follows by induction over n .

2. \mathcal{S}_{\circ}^* admits the given representation. First of all,

$$\mathcal{S}_{\circ}^* \mathbf{1} = \mathbf{1} = s_{\mathbf{1}}(\mathbf{1}) \mathbf{1} = \sum_{\eta \in \text{Splits}(\mathbf{1})} s_{\zeta}(\eta) \eta.$$

Assuming $\mathcal{S}_\circ^* \zeta$ admits the representation (3.25) for some $\zeta \in \mathcal{F}_I$, we get

$$\begin{aligned}
\mathcal{S}_\circ^* [\zeta]_i &= -[\zeta]_i - m_\circ(\text{id} \otimes \mathcal{S}_\circ^*) \tilde{\Delta}_* [\zeta]_i = - \sum_{\substack{(C,T) \in \text{Cuts}([\zeta]_i): \\ T \neq [\zeta]_i}} c_{[\zeta]_i}(C,T) C \odot \mathcal{S}_\circ^* T \\
&= - \sum_{\substack{(C,T) \in \text{Cuts}([\zeta]_i): \\ T \neq [\zeta]_i}} \sum_{S \in \text{Splits}(T)} s_T(S) c_{[\zeta]_i}(C,T) CS \\
&= - \sum_{\bar{S} \in \text{Splits}([\zeta]_i)} \sum_{\substack{(C,T) \in \text{Cuts}([\zeta]_i), S \in \text{Splits}(T): \\ T \neq [\zeta]_i, CS = \bar{S}}} s_T(S) c_{[\zeta]_i}(C,T) CS \\
&= \sum_{\bar{S} \in \text{Splits}([\zeta]_i)} s_{[\zeta]_i}(\bar{S}) \bar{S}.
\end{aligned}$$

Assuming $\mathcal{S}_\circ^* \zeta_1$ and $\mathcal{S}_\circ^* \zeta_2$ admit the representation (3.25) for some $\zeta_1, \zeta_2 \in \mathcal{F}_I$, we get

$$\begin{aligned}
\mathcal{S}_\circ^*(\zeta_1 \zeta_2) &= \mathcal{S}_\circ^* \zeta_1 \odot \mathcal{S}_\circ^* \zeta_2 = \sum_{S_1 \in \text{Splits}(\zeta_1)} \sum_{S_2 \in \text{Splits}(\zeta_2)} s_{\zeta_1}(S_1) s_{\zeta_2}(S_2) S_1 S_2 \\
&= \sum_{S \in \text{Splits}(\zeta_1 \zeta_2)} \sum_{\substack{S_1 \in \text{Splits}(\zeta_1), S_2 \in \text{Splits}(\zeta_2): \\ S_1 S_2 = S}} s_{\zeta_1}(S_1) s_{\zeta_2}(S_2) S_1 S_2 = \\
&= \sum_{S \in \text{Splits}(\zeta_1 \zeta_2)} \sum_{\substack{S_1 \in \text{Splits}(\zeta_1), S_2 \in \text{Splits}(\zeta_2): \\ S_1 S_2 = S}} s_{\zeta_1}(S_1) s_{\zeta_2}(S_2) S = \sum_{S \in \text{Splits}(\zeta_1 \zeta_2)} s_{\zeta_1 \zeta_2}(S) S.
\end{aligned}$$

□

For a tree τ , each forest η , except τ itself, in the set $\text{Splits}(\tau)$ is achieved by multiplying all the forests resulting from a finite sequence of consecutive admissible cuts of τ . The coefficient $s_\tau(\eta)$ then gives the number of possibilities to draw such sequences of odd length into a fixed planar drawing of the tree τ , minus the number of possibilities with even length, where the order of the cuts in the sequence doesn't count, each cut may only appear once in the sequence and empty as well as full cuts are excluded. For the special case $\tau \in \text{Splits}(\tau)$, we have $s_\tau(\tau) = -1$.

3.5.3 Cocommutative dual Hopf algebra

Definition 3.53. Let $\star : \langle \mathcal{F}_I \rangle \times \langle \mathcal{F}_I \rangle$ be the bilinear map generated by

$$C \star T := \sum_{\substack{\zeta \in \mathcal{F}_I: \\ (C,T) \in \text{Cuts}(\zeta)}} c_\zeta(C,T) \zeta, \quad C, T \in \mathcal{F}_I,$$

furthermore $\Delta_\circ : \langle \mathcal{F}_I \rangle \rightarrow \langle \mathcal{F}_I \rangle \otimes \langle \mathcal{F}_I \rangle$ the linear map generated by

$$\Delta_\circ \zeta := \sum_{\substack{\zeta_1, \zeta_2 \in \mathcal{F}_I: \\ \zeta_1 \zeta_2 = \zeta}} \zeta_1 \otimes \zeta_2, \quad \zeta \in \mathcal{F}_I$$

and $\mathcal{S}_\star^\circ : \langle \mathcal{F}_I \rangle \rightarrow \langle \mathcal{F}_I \rangle$ the linear map generated by

$$\mathcal{S}_\star^\circ \eta := \sum_{\substack{\zeta \in \mathcal{F}_I: \\ \eta \in \text{Splits}(\zeta)}} s_\zeta(\eta) \zeta, \quad \zeta \in \mathcal{F}_I$$

Finally, let $\langle \cdot, \cdot \rangle : \langle \mathcal{F}_I \rangle \times \langle \mathcal{F}_I \rangle \rightarrow \mathbb{R}$ be bilinearly generated by

$$\langle \zeta, \eta \rangle := \delta_{\zeta, \eta}, \quad \zeta, \eta \in \mathcal{F}_I.$$

As noted in Section 2.2 [HK14], the product \star is often called *Grossman-Larson product*, where the name refers to the paper [GL89].

Theorem 3.54. (Based on Section 2.2 [HK14]) $(\langle \mathcal{F}_I \rangle, \star, \Delta_\odot, \mathcal{S}_\star^\odot)$ is the cocommutative dual Hopf algebra of $(\langle \mathcal{F}_I \rangle, \odot, \Delta_\star, \mathcal{S}_\odot^\star)$ under the duality pairing $\langle \cdot, \cdot \rangle$. It is connectedly graded by $(\langle \mathcal{F}_{I,n} \rangle)_n$, too.

Proof.

1. m_\star is the dual operator of Δ_\star . For all $C, T, \eta \in \mathcal{F}_I$, we have

$$\begin{aligned} \langle m_\star(C \otimes T), \eta \rangle &= \sum_{\substack{\zeta \in \mathcal{F}_I: \\ (C, T) \in \text{Cuts}(\zeta)}} c_\zeta(C, T) \delta_{\zeta, \eta} = \sum_{(\bar{C}, \bar{T}) \in \text{Cuts}(\eta)} c_\eta(C, T) \delta_{C, \bar{C}} \delta_{T, \bar{T}} \\ &= \sum_{(\bar{C}, \bar{T}) \in \text{Cuts}(\eta)} c_\eta(\bar{C}, \bar{T}) \delta_{C, \bar{C}} \delta_{T, \bar{T}} = \sum_{(\bar{C}, \bar{T}) \in \text{Cuts}(\eta)} c_\eta(\bar{C}, \bar{T}) \langle C \otimes T, \bar{C} \otimes \bar{T} \rangle \\ &= \langle C \otimes T, \Delta_\star \eta \rangle. \end{aligned}$$

2. Δ_\odot is the dual operator of m_\odot . For all $\zeta, \eta_1, \eta_2 \in \mathcal{F}_I$, we have

$$\langle \Delta_\odot \zeta, \eta_1 \otimes \eta_2 \rangle = \sum_{\substack{\zeta_1, \zeta_2 \in \mathcal{F}_I: \\ \zeta_1 \zeta_2 = \zeta}} \delta_{\zeta_1, \eta_1} \delta_{\zeta_2, \eta_2} = \delta_{\zeta, \eta_1 \eta_2} = \langle \zeta, \eta_1 \odot \eta_2 \rangle.$$

3. \mathcal{S}_\star^\odot is the dual operator of \mathcal{S}_\odot^\star . For all $\eta, \zeta \in \mathcal{F}_I$, we have

$$\langle \mathcal{S}_\star^\odot \eta, \zeta \rangle = \sum_{\substack{\zeta' \in \mathcal{F}_I: \\ \eta \in \text{Splits}(\zeta')}} s_{\zeta'}(\eta) \delta_{\zeta', \zeta} = \sum_{\eta' \in \text{Splits}(\zeta)} s_\zeta(\eta) \delta_{\eta', \eta} = \sum_{\eta' \in \text{Splits}(\zeta)} s_\zeta(\eta') \delta_{\eta', \eta} = \langle \eta, \mathcal{S}_\odot^\star \zeta \rangle.$$

4. $(\langle \mathcal{F}_I \rangle, \star, \Delta_\odot, \mathcal{S}_\star^\odot)$ is the cocommutative dual Hopf algebra of $(\langle \mathcal{F}_I \rangle, \odot, \Delta_\star, \mathcal{S}_\odot^\star)$. We obviously have that u is the dual operator of ε and vice versa. Since $(\langle \mathcal{F}_I \rangle, \odot, \Delta_\star, \mathcal{S}_\odot^\star)$ is a commutative Hopf algebra with unit map u and counit ε according to Theorem 3.49, we get from the first three steps using Theorem 2.26 that $(\langle \mathcal{F}_I \rangle, \star, \Delta_\odot, \mathcal{S}_\star^\odot)$ is its cocommutative dual Hopf algebra with unit map u and counit ε .
5. $(\langle \mathcal{F}_I \rangle, \star, \Delta_\odot, \mathcal{S}_\star^\odot)$ is graded. \star is graded by $(\langle \mathcal{F}_{I,n} \rangle)_n$ since $|C| + |T| = |\zeta|$ for all $\zeta \in \mathcal{F}_I$ and $(C, T) \in \text{Cuts}(\zeta)$. Δ_\odot is graded since $|\zeta_1| + |\zeta_2| = |\zeta_1 \zeta_2|$ for all $\zeta_1, \zeta_2 \in \mathcal{F}_I$. \mathcal{S}_\star^\odot is graded since $|S| = |\eta|$ for all $\eta \in \mathcal{F}_I$ and $S \in \text{Splits}(\eta)$.

□

Obviously, $\text{Prim}(\langle \mathcal{F}_I \rangle, \Delta_\odot) = \langle \mathcal{F}_I \rangle$. Hence, $\text{GrLk}(\langle \mathcal{F}_I \rangle_\Pi, \Delta_\odot) = \exp_\star \langle \mathcal{F}_I \rangle_\Pi$. As noted in Section 2.3 [HK14], this group is also called the *Butcher group*.

3.5.4 Homomorphisms between words and forests

Based on the terminologies used in [HK14] and in [Reu93], we call

$$\mathcal{W}_I := \{\mathbf{1}\} \cup \{e_{i_1 \dots i_n} \mid n \in \mathbb{N}, i_j \in I\}, \quad e_a \neq e_b \text{ if } a \neq b,$$

the set of *words* over the alphabet I , where $\mathbf{1}$ is called the *empty word*. Define $|\cdot| : \mathcal{W}_I \rightarrow \mathbb{N}_0$ via $|e_{i_1 \dots i_n}| := n$ and $|\mathbf{1}| := 0$. The number $|w|$ is called the *length* of the word w . Put $\mathcal{W}_I^n := \{w \in \mathcal{W}_I : |w| \leq n\}$. We may then identify $\mathbb{T}(\langle I \rangle)$ with $\langle \mathcal{W}_I \rangle$ and $\mathbb{T}^n(\langle I \rangle)$ with $\langle \mathcal{W}_I^n \rangle$ via $e_{i_1 \dots i_m} := e_{i_1} \dot{\otimes} \dots \dot{\otimes} e_{i_m}$. The duality pairing $\langle \cdot, \cdot \rangle : \mathcal{W}_I \times \mathcal{W}_I \rightarrow \mathbb{R}$ generated by

$$\langle w_1, w_2 \rangle := \delta_{w_1, w_2}, \quad w_1, w_2 \in \mathcal{W}_I$$

is then also an inner product.

Theorem 3.55. (Section 4.1 [HK14]) *The linear operator*

$$\phi : (\langle \mathcal{F}_I \rangle, \odot, \Delta_*, \mathcal{S}_\odot^*) \rightarrow (\langle \mathcal{W}_I \rangle, \sqcup, \Delta_{\dot{\otimes}}, \mathcal{S}_{\sqcup}^{\dot{\otimes}})$$

recursively generated by

$$\phi(\mathbf{1}) := \mathbf{1}, \quad \phi([\zeta]_i) := \phi(\zeta) \dot{\otimes} e_i, \quad \phi(\zeta_1 \zeta_2) := \phi(\zeta_1) \sqcup \phi(\zeta_2) \quad (3.26)$$

is a Hopf algebra homomorphism. The linear operator $\hat{\phi} : (\langle \mathcal{W}_I \rangle, \Delta_{\dot{\otimes}}) \rightarrow (\langle \mathcal{F}_I \rangle, \Delta_*)$ recursively generated by

$$\hat{\phi}(\mathbf{1}) := \mathbf{1}, \quad \hat{\phi}(w \dot{\otimes} e_i) := [\hat{\phi}(w)]_i$$

is a coalgebra monomorphism. We have $\phi \hat{\phi} = \text{id}$.

Proof. Putting $L_i x := x \dot{\otimes} e_i$ for all $x \in \langle \mathcal{W}_I \rangle$, we get directly from the definition (3.15) that

$$\Delta_{\dot{\otimes}} L_i w = \Delta_{\dot{\otimes}}(w \dot{\otimes} e_i) = (w \dot{\otimes} e_i) \otimes \mathbf{1} + \sum_{(w)}^{\dot{\otimes}} w_1 \otimes (w_2 \dot{\otimes} e_1) = L_i w \otimes \mathbf{1} + (\text{id} \otimes L_i) \Delta_{\dot{\otimes}} w.$$

Since $\phi \hat{\phi} \mathbf{1} = \mathbf{1}$ and $\phi \hat{\phi}(w \dot{\otimes} e_i) = \phi([\hat{\phi}(w)]_i) = \phi \hat{\phi}(w) \dot{\otimes} e_i$, we get $\phi \hat{\phi} = \text{id}$ inductively. In particular, this shows that $\hat{\phi}$ is injective.

1. ϕ is a Hopf algebra homomorphism. ϕ is obviously an algebra homomorphism by the third equation in (3.26). Also, we have $\phi u = u$ by definition and $\varepsilon \phi \zeta = \delta_{\phi \zeta, \mathbf{1}} = \delta_{\zeta, \mathbf{1}} = \varepsilon \zeta$ for all $\zeta \in \mathcal{F}_I$. By Theorem 2.14, it only remains to show that ϕ is a coalgebra homomorphism. For this, we first have

$$\Delta_{\dot{\otimes}} \phi(\mathbf{1}) = \Delta_{\dot{\otimes}} \mathbf{1} = \mathbf{1} \otimes \mathbf{1} = (\phi \otimes \phi)(\mathbf{1} \otimes \mathbf{1}) = (\phi \otimes \phi) \Delta_* \mathbf{1}.$$

Assuming we have $\Delta_{\dot{\otimes}} \phi(\zeta) = (\phi \otimes \phi) \Delta_* \zeta$ for some $\zeta \in \mathcal{F}_I$, we get

$$\begin{aligned} (\phi \otimes \phi) \Delta_* [\zeta]_i &= (\phi \otimes \phi)([\zeta]_i \otimes \mathbf{1}) + (\phi \otimes \phi)(\text{id} \otimes [\cdot]_i) \Delta_* \zeta \\ &= L_i \phi(\zeta) \otimes \mathbf{1} + (\text{id} \otimes L_i)(\phi \otimes \phi) \Delta_* \zeta = L_i \phi(\zeta) \otimes \mathbf{1} + (\text{id} \otimes L_i) \Delta_{\dot{\otimes}} \phi(\zeta) \\ &= \Delta_{\dot{\otimes}} L_i \phi(\zeta) = \Delta_{\dot{\otimes}} \phi([\zeta]_i). \end{aligned}$$

Assuming we have $\Delta_{\dot{\otimes}} \phi(\zeta_1) = (\phi \otimes \phi) \Delta_* \zeta_1$ and $\Delta_{\dot{\otimes}} \phi(\zeta_2) = (\phi \otimes \phi) \Delta_* \zeta_2$ for some $\zeta_1, \zeta_2 \in \mathcal{F}_I$, we get

$$\begin{aligned} \Delta_{\dot{\otimes}} \phi(\zeta_1 \zeta_2) &= \Delta_{\dot{\otimes}} \phi(\zeta_1) \sqcup \Delta_{\dot{\otimes}} \phi(\zeta_2) = (\phi \otimes \phi) \Delta_* \zeta_1 \sqcup (\phi \otimes \phi) \Delta_* \zeta_2 \\ &= (\phi \otimes \phi)(\Delta_* \zeta_1 \odot \Delta_* \zeta_2) = (\phi \otimes \phi) \Delta_* (\zeta_1 \zeta_2). \end{aligned}$$

2. $\hat{\phi}$ is a coalgebra homomorphism. As always, $\Delta_*\hat{\phi}(\mathbf{1}) = (\hat{\phi} \otimes \hat{\phi})\Delta_{\otimes}\mathbf{1}$ follows immediately. Assuming we have $\Delta_*\hat{\phi}(w) = (\hat{\phi} \otimes \hat{\phi})\Delta_{\otimes}w$ for some $w \in \mathcal{W}_I$, we get

$$\begin{aligned}\Delta_*\hat{\phi}(w \dot{\otimes} e_i) &= \Delta_*[\hat{\phi}(w)]_i = [\hat{\phi}(w)]_i \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i)\Delta_*\hat{\phi}(w) \\ &= \hat{\phi}(w \dot{\otimes} e_i) \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i)(\hat{\phi} \otimes \hat{\phi})\Delta_{\otimes}w \\ &= (\hat{\phi} \otimes \hat{\phi})(L_i w \otimes \mathbf{1}) + (\hat{\phi} \otimes \hat{\phi})(\text{id} \otimes L_i)\Delta_{\otimes}w = (\hat{\phi} \otimes \hat{\phi})\Delta_{\otimes}L_i w.\end{aligned}$$

□

Theorem 3.56. (Lemma 4.9. [HK14] and its proof)

The map $\psi : ((\mathcal{F}_I), \odot, \Delta_*, \mathcal{S}_{\odot}^*) \rightarrow (\mathbb{T}((\mathcal{F}_I)), \sqcup, \Delta_{\otimes}, \mathcal{S}_{\sqcup}^{\otimes})$ recursively linearly generated by

$$\psi(\mathbf{1}) := \mathbf{1}, \quad \psi([\zeta]_i) := m_{\otimes}(\psi \otimes [\cdot]_i)\Delta_*\zeta, \quad \psi(\zeta_1 \zeta_2) := \psi(\zeta_1) \sqcup \psi(\zeta_2) \quad (3.27)$$

is a Hopf algebra monomorphism.

Proof.

1. ψ is a Hopf algebra homomorphism. ψ is obviously an algebra homomorphism by the third equation in (3.27). We first have

$$\Delta_{\otimes}\psi(\mathbf{1}) = \Delta_{\otimes}\mathbf{1} = \mathbf{1} \otimes \mathbf{1} = (\psi \otimes \psi)(\mathbf{1} \otimes \mathbf{1}) = (\psi \otimes \psi)\Delta_*\mathbf{1}.$$

Assuming we have $\Delta_{\otimes}\psi(\zeta) = (\psi \otimes \psi)\Delta_*\zeta$ for some $\zeta \in \mathcal{F}_I$, we get

$$\begin{aligned}\Delta_{\otimes}\psi([\zeta]_i) &= \Delta_{\otimes}m_{\otimes}(\psi \otimes [\cdot]_i)\Delta_*\zeta = m_{\otimes}(\Delta_{\otimes} \otimes \text{id})(\psi \otimes [\cdot]_i)\Delta_*\zeta + \psi([\zeta]_i) \otimes \mathbf{1} \\ &= m_{\otimes}(\psi \otimes \psi \otimes \text{id} \otimes [\cdot]_i)(\Delta_* \otimes \text{id})\Delta_*\zeta + \psi([\zeta]_i) \otimes \mathbf{1} \\ &= m_{\otimes}(\psi \otimes \psi \otimes \text{id} \otimes [\cdot]_i)(\text{id} \otimes \Delta_*)\Delta_*\zeta + \psi([\zeta]_i) \otimes \mathbf{1} \\ &= (\psi \otimes m_{\otimes})(\text{id} \otimes \psi \otimes [\cdot]_i)(\text{id} \otimes \Delta_*)\Delta_*\zeta + \psi([\zeta]_i) \otimes \mathbf{1} \\ &= (\psi \otimes \psi)(\text{id} \otimes [\cdot]_i)\Delta_*\zeta + \psi([\zeta]_i) \otimes \mathbf{1} \\ &= (\psi \otimes \psi)\Delta_*[\zeta]_i.\end{aligned}$$

Assuming we have $\Delta_{\otimes}\psi(\zeta_1) = (\psi \otimes \psi)\Delta_*\zeta_1$ and $\Delta_{\otimes}\psi(\zeta_2) = (\psi \otimes \psi)\Delta_*\zeta_2$ for some $\zeta_1, \zeta_2 \in \mathcal{F}_I$, we get

$$\begin{aligned}\Delta_{\otimes}\psi(\zeta_1 \odot \zeta_2) &= \Delta_{\otimes}\psi(\zeta_1) \sqcup \Delta_{\otimes}\psi(\zeta_2) = (\psi \otimes \psi)\Delta_*\zeta_1 \sqcup (\psi \otimes \psi)\Delta_*\zeta_2 \\ &= (\psi \otimes \psi)(\Delta_*\zeta_1 \odot \Delta_*\zeta_2) = (\psi \otimes \psi)\Delta_*(\zeta_1 \odot \zeta_2).\end{aligned}$$

2. ψ is injective. See the proof of Lemma 4.9. [HK14].

□

Chapter 4

Rough paths

4.1 Geometric rough paths

We shortly write $\mathcal{W}_d := \mathcal{W}_{\{1, \dots, d\}}$ for all $d \in \mathbb{N}$. Define

$$\mathbf{G}_d := \text{GrLk}(\mathbb{T}(\mathbb{R}^d), \Delta_\omega) = \{g \in \mathbb{T}(\mathbb{R}^d) \mid \Delta_\omega g = g \otimes g\} = \exp_{\dot{\otimes}}(\mathbf{L}_d),$$

where

$$\mathbf{L}_d := \text{Prim}(\mathbb{T}(\mathbb{R}^d), \Delta_\omega) = \{l \in \mathbb{T}(\mathbb{R}^d) \mid \Delta_\omega l = \mathbf{1} \otimes l + l \otimes \mathbf{1}\} = (\mathbb{R}^d)_{\mathfrak{b}_{\dot{\otimes}}},$$

as well as

$$\mathbf{G}_d^n := \text{GrLk}^n(\mathbb{T}(\mathbb{R}^d), \Delta_\omega) = \{g \in \mathbb{T}^n(\mathbb{R}^d) \mid \Delta_\omega g = \pi^n(g \otimes g)\} = \pi^n \mathbf{G}_d = \exp_{\dot{\otimes}_n}(\mathbf{L}_d^n),$$

where

$$\mathbf{L}_d^n := \text{Prim}(\mathbb{T}^n(\mathbb{R}^d), \Delta_\omega) = \{l \in \mathbb{T}^n(\mathbb{R}^d) \mid \Delta_\omega l = \mathbf{1} \otimes l + l \otimes \mathbf{1}\} = \pi^n (\mathbb{R}^d)_{\mathfrak{b}_{\dot{\otimes}}}.$$

Define $\|\cdot\|_{\mathbf{G}_d^n} : \mathbf{G}_d^n \rightarrow [0, \infty)$ by (Section 4 [HK14])

$$\|g\|_{\mathbf{G}_d^n} := \sum_{m=1}^n \|\pi_m \log_{\dot{\otimes}_n}(g)\|^{1/m},$$

where $\|l\| := \sqrt{\langle l, l \rangle}$. Put $\|\cdot\|_n := \|\pi_n \cdot\|_n$.

Definition 4.1. (Equation (4.3) and Definition 4.1. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder *weakly geometric rough path* is a map $\mathbf{X} : [0, T] \rightarrow \mathbf{G}_d^n$ with $\mathbf{X}_0 := \mathbf{1}$ such that

$$\sup_{s < t} \frac{\|\mathbf{X}_{st}\|_{\mathbf{G}_d^n}}{|t - s|^\gamma} < \infty,$$

where $\mathbf{X}_{st} := \mathbf{X}_s^{\dot{\otimes}_n} \dot{\otimes}_n \mathbf{X}_t = \mathcal{S}_{\dot{\otimes}}^\omega \mathbf{X}_s \dot{\otimes}_n \mathbf{X}_t$ and n is the integer part of $\frac{1}{\gamma}$. The set of all such maps $\mathbf{X} : [0, T] \rightarrow \mathbf{G}_d^n$ is denoted by $\mathbf{R}^\gamma([0, T], \mathbb{R}^d)$.

Remark 4.2. The word 'weakly' comes from the fact that there is a notion of geometric rough paths which is a bit stronger. For details on this distinction, see [FV06].

We present three more definitions which will turn out to be equivalent.

Definition 4.2.1. (Based on Equation (4.3) and Definition 4.1. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder *weakly geometric rough path* is a map $\bar{\mathbf{X}} : [0, T] \rightarrow G_d$ with $\bar{\mathbf{X}}_0 := \mathbf{1}$ such that

$$\sup_{s < t} \frac{\|\pi^n \bar{\mathbf{X}}_{st}\|_{G_d^n}}{|t-s|^\gamma} < \infty \quad \forall n \in \mathbb{N},$$

where $\bar{\mathbf{X}}_{st} := \bar{\mathbf{X}}_s^{\otimes -1} \otimes \bar{\mathbf{X}}_t = \mathcal{S}_{\otimes}^w \bar{\mathbf{X}}_s \otimes \bar{\mathbf{X}}_t$. The set of all such maps $\bar{\mathbf{X}} : [0, T] \rightarrow G_d$ is denoted by $\bar{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d)$.

Definition 4.2.2. (Definition 1.2. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder *weakly geometric rough path* is a map $\check{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{W}_d^n \rangle^*$ such that

1. $\check{\mathbf{X}}_{st}(\mathbf{1}) = 1$ and $\check{\mathbf{X}}_{st}(w_1 \sqcup w_2) = \check{\mathbf{X}}_{st}(w_1)\check{\mathbf{X}}_{st}(w_2)$ for all words $w_1, w_2 \in \mathcal{W}_d^n$ such that $|w_1| + |w_2| \leq n$,
2. $\check{\mathbf{X}}_{tt} = \varepsilon_n$ and $\check{\mathbf{X}}_{st}(w) = (\check{\mathbf{X}}_{su} \otimes \check{\mathbf{X}}_{ut})\Delta_{\otimes} w = \sum_{(w)}^{\otimes} \check{\mathbf{X}}_{su}(w^1)\check{\mathbf{X}}_{ut}(w^2)$ for all words $w \in \mathcal{W}_d^n$,
3. $\sup_{s \neq t} \frac{|\check{\mathbf{X}}_{st}(w)|}{|t-s|^{\gamma|w|}} < \infty$ for all words $w \in \mathcal{W}_d^n$,

where n is the integer part of $\frac{1}{\gamma}$. Put $\check{\mathbf{X}}_t := \check{\mathbf{X}}_{0t}$. The set of all such maps $\check{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{W}_d^n \rangle^*$ is denoted by $\check{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d)$.

Definition 4.2.3. (Based on Definition 1.2. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder *weakly geometric rough path* is a map $\hat{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{W}_d \rangle^*$ such that

1. $\hat{\mathbf{X}}_{st}(\mathbf{1}) = 1$ and $\hat{\mathbf{X}}_{st}(w_1 \sqcup w_2) = \hat{\mathbf{X}}_{st}(w_1)\hat{\mathbf{X}}_{st}(w_2)$ for all words $w_1, w_2 \in \mathcal{W}_d$,
2. $\hat{\mathbf{X}}_{tt} = \varepsilon$ and $\hat{\mathbf{X}}_{st}(w) = (\hat{\mathbf{X}}_{su} \otimes \hat{\mathbf{X}}_{ut})\Delta_{\otimes} w = \sum_{(w)}^{\otimes} \hat{\mathbf{X}}_{su}(w^1)\hat{\mathbf{X}}_{ut}(w^2)$ for all words $w \in \mathcal{W}_d$,
3. $\sup_{s \neq t} \frac{|\hat{\mathbf{X}}_{st}(w)|}{|t-s|^{\gamma|w|}} < \infty$ for all words $w \in \mathcal{W}_d$.

Put $\hat{\mathbf{X}}_t := \hat{\mathbf{X}}_{0t}$. The set of all such maps $\hat{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{W}_d \rangle^*$ is denoted by $\hat{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d)$.

Theorem 4.3. Lyons' Extension Theorem (Theorem 3.7 [LCL07]) *Let $\gamma \in (0, 1)$ and n be the integer part of γ^{-1} . Let $X : [0, T]^2 \rightarrow \mathbb{T}^n(\mathbb{R}^d)$ be such that $\pi_0 X_{st} = 1$ for all $s, t \in [0, T]$, such that*

$$X_{su} \dot{\otimes}_n X_{ut} = X_{st} \quad \forall s, u, t \in [0, T]$$

and such that

$$\sup_{s < t} \frac{\|X_{su}\|_m}{|u-s|^{m\gamma}} < \infty \quad \forall m \in \{0, \dots, n\}. \quad (4.1)$$

Then, there is a unique extension $\bar{X} : [0, T]^2 \rightarrow \mathbb{T}(\mathbb{R}^d)$ such that $\pi^n \bar{X}_{st} = X_{st}$ for all $s, t \in [0, T]$, such that

$$\bar{X}_{su} \dot{\otimes} \bar{X}_{ut} = \bar{X}_{st} \quad \forall s, u, t \in [0, T] \quad (4.2)$$

and such that

$$\sup_{s < t} \frac{\|\bar{X}_{su}\|_m}{|u-s|^{m\gamma}} < \infty \quad \forall m \in \mathbb{N}_0.$$

Theorem 4.4. (Corollary 3.9 [CDLL15]) *In the context of Theorem 4.3, if $X_{st} \in G_d^n$ for all $s, t \in [0, T]^2$, then $\bar{X}_{st} \in G_d$ for all $s, t \in [0, T]^2$.*

Note that for our setting of paths in \mathbb{R}^d , this result is actually known for several years already, but in [CDLL15], it is shown for the more general case of paths in a real Banach space.

Lemma 4.5. (Based on Proposition 4 [LV07])

1. For all $k \in \mathbb{N}$, there is a constant $C_k > 0$ such that

$$|\langle w, g \rangle| \leq C_{|w|} \|g\|_{\mathbb{G}_d^n}^{|w|} \quad \forall n \in \mathbb{N}, g \in \mathbb{G}_d^n, w \in \mathcal{W}_d^n \setminus \{\mathbf{1}\}.$$

2. For all $n \in \mathbb{N}$, there is a constant $C'_n > 0$ such that

$$\|g\|_{\mathbb{G}_d^n} \leq C'_n \sup_{w \in \mathcal{W}_d^n \setminus \{\mathbf{1}\}} |\langle w, g \rangle|^{1/|w|} \quad \forall g \in \mathbb{G}_d^n.$$

Proof. (Based on the proof of Proposition 4 [LV07])

1. Let $n \in \mathbb{N}_0$ and $x \in L_d^n$ be arbitrary. For every $m, k \in \mathbb{N}$ with $m \leq n$, we have

$$\begin{aligned} \|x^{\otimes_n m}\|_k &= \left\| \left(\sum_{v \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}} v \langle v, x \rangle \right)^m \right\|_k \\ &= \left\| \sum_{\substack{v_1, \dots, v_m \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}: \\ |v_1| + \dots + |v_m| = k}} \langle v_1, x \rangle \cdots \langle v_m, x \rangle v_1 \otimes \cdots \otimes v_m \right\|_k \\ &\leq \sum_{\substack{v_1, \dots, v_m \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}: \\ |v_1| + \dots + |v_m| = k}} \prod_{i=1}^m |v_i, x| \leq N_{m,k} \sup_{\substack{v_1, \dots, v_m \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}: \\ |v_1| + \dots + |v_m| = k}} \prod_{i=1}^m |v_i, x| \\ &\leq N_{m,k} \sup_{\substack{l_1, \dots, l_m \in \mathbb{N}: \\ l_1 + \dots + l_m = k}} \prod_{i=1}^m \|x\|_{l_i} \\ &\leq N_{m,k} \sup_{\substack{l_1, \dots, l_m \in \mathbb{N}: \\ l_1 + \dots + l_m = k}} \prod_{i=1}^m \|\exp_{\otimes_n}(x)\|_{\mathbb{G}_d^n}^{l_i} = N_{m,k} \|\exp_{\otimes_n}(x)\|_{\mathbb{G}_d^n}^k, \end{aligned}$$

where $N_{m,k} := |\{v_1, \dots, v_m \in \mathcal{W}_d^k \setminus \{\mathbf{1}\} \mid |v_1| + \dots + |v_m| = k\}| < \infty$. Thus, for any $w \in \mathcal{W}_d^n$, we get

$$\begin{aligned} |\langle w, \exp_{\otimes_n}(x) \rangle| &= \left| \sum_{m=0}^n \frac{1}{m!} \langle w, x^{\otimes_n m} \rangle \right| = \left| \sum_{m=0}^{|w|} \frac{1}{m!} \langle w, \pi_{|w|} x^{\otimes_n m} \rangle \right| \\ &\leq \sum_{m=0}^{|w|} \frac{1}{m!} \|x^{\otimes_n m}\|_{|w|} \leq \underbrace{\sum_{m=0}^{|w|} \frac{N_{m,|w|}}{m!}}_{:=C_{|w|}} \|\exp_{\otimes_n}(x)\|_{\mathbb{G}_d^n}^{|w|}. \end{aligned}$$

2. Let $n \in \mathbb{N}_0$ and $\mathbf{1} + u \in \mathbb{G}_d^n$ be arbitrary. For every $m, k \in \mathbb{N}$ with $m \leq n$, we have, with

$N_{m,k}$ and similar first steps as above,

$$\begin{aligned} \|u^m\|_k^{1/k} &\leq N_{m,k}^{1/k} \sup_{\substack{v_1, \dots, v_m \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}: \\ |v_1| + \dots + |v_m| = k}} \prod_{i=1}^m |v_i, u|^{1/k} \\ &\leq N_{m,k}^{1/k} \sup_{\substack{v_1, \dots, v_m \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}: \\ |v_1| + \dots + |v_m| = k}} \prod_{i=1}^m \left(\sup_{v \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}} |\langle v, u \rangle|^{1/|v|} \right)^{|v_i|/k} = N_{m,k}^{1/k} \sup_{v \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}} |\langle v, u \rangle|^{1/|v|}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathbf{1} + u\|_{G_d^n} &= \sum_{k=1}^n \|\log_{\otimes_n}(\mathbf{1} + u)\|_k^{1/k} = \sum_{k=1}^n \left\| \sum_{m=1}^k \frac{(-1)^{m-1} u^{\otimes_n m}}{m} \right\|_k^{1/k} \\ &\leq \sum_{k=1}^n \left(\sum_{m=1}^k \frac{1}{m} \|u^{\otimes_n m}\|_k \right)^{1/k} \stackrel{(\diamond)}{\leq} \sum_{k=1}^n \sum_{m=1}^k \frac{1}{\sqrt[k]{m}} \|u^{\otimes_n m}\|_k^{1/k} \\ &\leq \underbrace{\sum_{k=1}^n \sum_{m=1}^k \sqrt[k]{\frac{N_{m,k}}{m}}}_{:= C'_n} \sup_{v \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}} |\langle v, u \rangle|^{1/|v|} = C'_n \sup_{v \in \mathcal{W}_d^k \setminus \{\mathbf{1}\}} |\langle v, \mathbf{1} + u \rangle|^{1/|v|}, \end{aligned}$$

where at (\diamond) , we iteratively used the simple fact that $(a+b)^{1/k} \leq a^{1/k} + b^{1/k}$ for all $a, b > 0$, $k \in \mathbb{N}$. □

The following theorem shows that the four definitions are indeed equivalent.

Theorem 4.6. *Let $\gamma \in (0, 1)$ and \mathbf{n} denote the integer part of $\frac{1}{\gamma}$. The maps*

$$\begin{aligned} I_1^\gamma &: \bar{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}^\gamma([0, T], \mathbb{R}^d), \bar{\mathbf{X}} \mapsto \mathbf{X}, \mathbf{X}_t := \pi^{\mathbf{n}} \bar{\mathbf{X}}_t, \\ I_2^\gamma &: \mathbb{R}^\gamma([0, T], \mathbb{R}^d) \rightarrow \check{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d), \mathbf{X} \mapsto \check{\mathbf{X}}, \check{\mathbf{X}}_{st}(w) := \langle w, \mathbf{X}_{st} \rangle, \\ I_3^\gamma &: \bar{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d) \rightarrow \hat{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d), \bar{\mathbf{X}} \mapsto \hat{\mathbf{X}}, \hat{\mathbf{X}}_{st}(w) := \langle w, \bar{\mathbf{X}}_{st} \rangle \end{aligned}$$

are well-defined and bijective.

Proof.

1. $I_1^\gamma : \bar{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}^\gamma([0, T], \mathbb{R}^d)$ is well-defined. For each $\bar{\mathbf{X}} \in \bar{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d)$, we have that $\mathbf{X} := \pi^{\mathbf{n}} \bar{\mathbf{X}} \in \mathbb{R}^\gamma([0, T], \mathbb{R}^d)$ since $\pi^{\mathbf{n}} G_d = G_d^{\mathbf{n}}$ (see Corollary 3.40) and

$$\pi^{\mathbf{n}}(\bar{\mathbf{X}}_{st}) = \pi^{\mathbf{n}}(\mathcal{S}_{\otimes}^{\mathbb{w}} \bar{\mathbf{X}}_s \dot{\otimes} \bar{\mathbf{X}}_t) = \pi^{\mathbf{n}} \mathcal{S}_{\otimes}^{\mathbb{w}} \bar{\mathbf{X}}_s \dot{\otimes}_{\mathbf{n}} \pi^{\mathbf{n}} \bar{\mathbf{X}}_t = \mathcal{S}_{\otimes}^{\mathbb{w}} \pi^{\mathbf{n}} \bar{\mathbf{X}}_s \dot{\otimes}_{\mathbf{n}} \pi^{\mathbf{n}} \bar{\mathbf{X}}_t = \mathcal{S}_{\otimes}^{\mathbb{w}} \mathbf{X}_s \dot{\otimes}_{\mathbf{n}} \mathbf{X}_t,$$

where the latter together with the analytic condition of Definition 4.2.1 implies the analytic condition of Definition 4.1.

2. I_1^γ is bijective. Direct consequence of Theorems 4.3 and 4.4. This is due to the fact that the analytic condition of Definition 4.1 implies (4.1) and that (4.2) implies the analytic condition of Definition 4.2.1. Indeed, by Lemma 4.5, we have

$$\|\mathbf{X}_{st}\|_m = \sqrt{\sum_{w \in \mathcal{W}_{d,m}} \langle w, \mathbf{X}_{st} \rangle^2} \leq \sqrt{|\mathcal{W}_{d,m}|} C_m \|\mathbf{X}_{st}\|_{G_d^m}$$

and

$$\|\bar{\mathbf{X}}_{st}\|_{\mathbf{G}_d^n}^m \leq (C'_m)^m \sup_{w \in \mathcal{W}_d^n \setminus \{\mathbf{1}\}} |\langle w, \bar{\mathbf{X}}_{st} \rangle| \leq (C'_m)^m \|\bar{\mathbf{X}}_{st}\|_m.$$

3. $I_2^\gamma : \mathbb{R}^\gamma([0, T], \mathbb{R}^d) \rightarrow \check{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d)$ is well-defined. Let $\mathbf{X} \in \mathbb{R}^\gamma([0, T], \mathbb{R}^d)$ be arbitrary. Then, $\check{\mathbf{X}}_{st} := \langle \cdot, \mathbf{X}_{st} \rangle \in \langle \mathcal{W}_d^n \rangle^*$ for all $s, t \in [0, T]$. We have $\check{\mathbf{X}}_{st}(\mathbf{1}) = \langle \mathbf{1}, \mathbf{X}_{st} \rangle = 1$ since $\mathbf{X}_{st} \in \mathbf{G}_d^n \subseteq \mathcal{G}_{\langle \mathcal{W}_d^n \rangle}$. For all $w_1, w_2 \in \mathcal{W}_d^n$ with $|w_1| + |w_2| \leq \mathbf{n}$ we get, using the duality of \sqcup and Δ_\sqcup ,

$$\begin{aligned} \check{\mathbf{X}}_{st}(w_1 \sqcup w_2) &= \langle w_1 \sqcup w_2, \mathbf{X}_{st} \rangle = \langle w_1 \otimes w_2, \Delta_\sqcup \mathbf{X}_{st} \rangle \stackrel{(\diamond)}{=} \langle w_1 \otimes w_2, \underline{\pi}^{\mathbf{n}}(\mathbf{X}_{st} \otimes \mathbf{X}_{st}) \rangle \\ &= \langle w_1 \otimes w_2, \mathbf{X}_{st} \otimes \mathbf{X}_{st} \rangle = \langle w_1, \mathbf{X}_{st} \rangle \langle w_2, \mathbf{X}_{st} \rangle = \check{\mathbf{X}}_{st}(w_1) \check{\mathbf{X}}_{st}(w_2), \end{aligned}$$

where at (\diamond) we used the fact that $\langle w_1 \otimes w_2, w_3 \otimes w_4 \rangle = 0$ for all $w_3, w_4 \in \mathcal{W}_d$ such that $|w_3| + |w_4| > \mathbf{n}$. Furthermore, using the duality of $\dot{\otimes}$ and $\Delta_{\dot{\otimes}}$, we have for all $w \in \mathcal{W}_d^n$ that

$$\begin{aligned} \check{\mathbf{X}}_{st}(w) &= \langle w, \mathbf{X}_{st} \rangle = \langle w, \mathbf{X}_{su} \dot{\otimes}_{\mathbf{n}} \mathbf{X}_{ut} \rangle \stackrel{(\diamond)}{=} \langle w, \pi^{\mathbf{n}}(\mathbf{X}_{su} \dot{\otimes} \mathbf{X}_{ut}) \rangle = \langle w, \mathbf{X}_{su} \dot{\otimes} \mathbf{X}_{ut} \rangle \\ &= \langle \Delta_{\dot{\otimes}} w, \mathbf{X}_{su} \otimes \mathbf{X}_{ut} \rangle = (\check{\mathbf{X}}_{su} \otimes \check{\mathbf{X}}_{ut}) \Delta_{\dot{\otimes}} w, \end{aligned}$$

where at (\diamond) we used the fact that $\langle w, v \rangle = 0$ for all $v \in \mathcal{W}_d$ such that $|v| > \mathbf{n}$. Finally, the fact that the analytic condition of Definition 4.2.2 follows from that of Definition 4.1 is a direct consequence of Lemma 4.5. Thus, $\check{\mathbf{X}} \in \check{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d)$.

4. I_2^γ is bijective. First of all, if $\bar{\mathbf{X}}^1, \bar{\mathbf{X}}^2 \in \bar{\mathbb{R}}^\gamma([0, T], \mathbb{R}^d)$ with $\bar{\mathbf{X}}^1 \neq \bar{\mathbf{X}}^2$, then there is $t \in [0, T]$ such that $\bar{\mathbf{X}}_t^1 \neq \bar{\mathbf{X}}_t^2$ and thus, there is $w \in \mathcal{W}_d$ such that $\langle w, \bar{\mathbf{X}}_{0t}^1 \rangle \neq \langle w, \bar{\mathbf{X}}_{0t}^2 \rangle$. Thus, I_2^γ is injective.

Let now $\check{\mathbf{X}} \in \mathbb{R}^\gamma([0, T], \mathbb{R}^d)$ be arbitrary. Put $\mathbf{X}_{st} := \sum_{w \in \mathcal{W}_d^n} \check{\mathbf{X}}(w)w$. Then, we obviously have $\check{\mathbf{X}}_{st} = \langle \cdot, \mathbf{X}_{st} \rangle$ for all $s, t \in [0, T]$. For all $w_1, w_2 \in \mathcal{W}_d^n$ such that $|w_1| + |w_2| \leq \mathbf{n}$, we get, again by the duality of \sqcup and Δ_\sqcup ,

$$\begin{aligned} \langle w_1 \otimes w_2, \Delta_\sqcup \mathbf{X}_{st} \rangle &= \langle w_1 \sqcup w_2, \mathbf{X}_{st} \rangle = \check{\mathbf{X}}_{st}(w_1 \sqcup w_2) = \check{\mathbf{X}}_{st}(w_1) \check{\mathbf{X}}_{st}(w_2) \\ &= \langle w_1, \mathbf{X}_{st} \rangle \langle w_2, \mathbf{X}_{st} \rangle = \langle w_1 \otimes w_2, \mathbf{X}_{st} \otimes \mathbf{X}_{st} \rangle \\ &= \langle w_1 \otimes w_2, \underline{\pi}^{\mathbf{n}}(\mathbf{X}_{st} \otimes \mathbf{X}_{st}) \rangle. \end{aligned}$$

Since $\langle \cdot, \cdot \rangle$ on $\langle \mathcal{W}_d \rangle \otimes \langle \mathcal{W}_d \rangle$ restricted to $[\langle \mathcal{W}_d \rangle \otimes \langle \mathcal{W}_d \rangle]_{\mathbf{n}}$ is again an inner product, we conclude $\Delta_\sqcup \mathbf{X}_{st} = \underline{\pi}^{\mathbf{n}}(\mathbf{X}_{st} \otimes \mathbf{X}_{st})$ and together with $\langle \mathbf{1}, \mathbf{X}_{st} \rangle = \check{\mathbf{X}}_{st}(\mathbf{1}) = 1$ therefore $\mathbf{X}_{st} \in \mathbf{G}_d^n$ for all $s, t \in [0, T]$. Furthermore, for all $w \in \mathcal{W}_d^n$ and all $s, u, t \in [0, T]$, we have

$$\begin{aligned} \langle w, \mathbf{X}_{st} \rangle &= \check{\mathbf{X}}_{st}(w) = (\check{\mathbf{X}}_{su} \otimes \check{\mathbf{X}}_{ut}) \Delta_{\dot{\otimes}} w = \langle \Delta_{\dot{\otimes}} w, \mathbf{X}_{su} \otimes \mathbf{X}_{ut} \rangle = \langle w, \mathbf{X}_{su} \dot{\otimes} \mathbf{X}_{ut} \rangle \\ &= \langle w, \pi^{\mathbf{n}}(\mathbf{X}_{su} \dot{\otimes} \mathbf{X}_{ut}) \rangle = \langle w, \mathbf{X}_{su} \dot{\otimes}_{\mathbf{n}} \mathbf{X}_{ut} \rangle. \end{aligned}$$

Since $\langle \cdot, \cdot \rangle$ on $\langle \mathcal{W}_d \rangle$ restricted to $\langle \mathcal{W}_d^n \rangle$ is again an inner product, we conclude $\mathbf{X}_{st} = \mathbf{X}_{su} \dot{\otimes}_{\mathbf{n}} \mathbf{X}_{ut}$. Thus, putting $\mathbf{X}_t := \mathbf{X}_{0t}$ for all $t \in [0, T]$ and using $\check{\mathbf{X}}_{00} = \varepsilon_{\mathbf{n}}$, we get

$$\mathbf{1} = \mathbf{X}_{00} = \mathbf{X}_{0t} \dot{\otimes}_{\mathbf{n}} \mathbf{X}_{t0} = \mathbf{X}_t \dot{\otimes}_{\mathbf{n}} \mathbf{X}_{t0}$$

and therefore $\mathbf{X}_{st} = \mathbf{X}_{s0} \dot{\otimes}_{\mathbf{n}} \mathbf{X}_{0t} = \mathbf{X}_s^{\dot{\otimes}_{\mathbf{n}}^{-1}} \dot{\otimes}_{\mathbf{n}} \mathbf{X}_t$. Hence, using Lemma 4.5 and the analytic condition of Definition 4.2.2, we get the analytic condition of Definition 4.1 and conclude that indeed $\mathbf{X} \in \mathbb{R}^\gamma([0, T], \mathbb{R}^d)$. This means we have shown that I_2^γ is also surjective.

5. $I_1^\gamma : \bar{R}^\gamma([0, T], \mathbb{R}^d) \rightarrow \hat{R}^\gamma([0, T], \mathbb{R}^d)$ is well-defined. Let $\mathbf{X} \in \bar{R}^\gamma([0, T], \mathbb{R}^d)$ be arbitrary. Then, $\hat{\mathbf{X}}_{st} := \langle \cdot, \bar{\mathbf{X}}_{st} \rangle \in \langle \mathcal{W}_d \rangle^*$ for all $s, t \in [0, T]$. We have $\hat{\mathbf{X}}_{st}(\mathbf{1}) = \langle \mathbf{1}, \bar{\mathbf{X}}_{st} \rangle = 1$ since $\bar{\mathbf{X}}_{st} \in G_d \subseteq \mathcal{G}(\mathcal{W}_d)$. For all $m \in \mathbb{N}$ and $w_1, w_2 \in \mathcal{W}_d^m$ with $|w_1| + |w_2| \leq m$ we get, using the duality of \sqcup and Δ_\sqcup ,

$$\begin{aligned} \hat{\mathbf{X}}_{st}(w_1 \sqcup w_2) &= \langle w_1 \sqcup w_2, \bar{\mathbf{X}}_{st} \rangle = \langle w_1 \sqcup w_2, \pi^m \bar{\mathbf{X}}_{st} \rangle = \langle w_1 \otimes w_2, \Delta_\sqcup \pi^m \bar{\mathbf{X}}_{st} \rangle \\ &= \langle w_1 \otimes w_2, \pi^m \Delta_\sqcup \bar{\mathbf{X}}_{st} \rangle = \langle w_1 \otimes w_2, \pi^m (\bar{\mathbf{X}}_{st} \otimes \bar{\mathbf{X}}_{st}) \rangle \\ &= \langle w_1 \otimes w_2, \bar{\mathbf{X}}_{st} \otimes \bar{\mathbf{X}}_{st} \rangle = \langle w_1, \bar{\mathbf{X}}_{st} \rangle \langle w_2, \bar{\mathbf{X}}_{st} \rangle = \hat{\mathbf{X}}_{st}(w_1) \hat{\mathbf{X}}_{st}(w_2). \end{aligned}$$

Furthermore, using the duality of $\dot{\otimes}$ and $\Delta_{\dot{\otimes}}$, we have for all $m \in \mathbb{N}$ and $w \in \mathcal{W}_d^m$ that

$$\begin{aligned} \hat{\mathbf{X}}_{st}(w) &= \langle w, \bar{\mathbf{X}}_{st} \rangle = \langle w, \bar{\mathbf{X}}_{su} \dot{\otimes} \bar{\mathbf{X}}_{ut} \rangle = \langle w, \pi^m (\bar{\mathbf{X}}_{su} \dot{\otimes} \bar{\mathbf{X}}_{ut}) \rangle = \langle w, \pi^m \bar{\mathbf{X}}_{su} \dot{\otimes}_m \pi^m \bar{\mathbf{X}}_{ut} \rangle \\ &= \langle w, \pi^m (\pi^m \bar{\mathbf{X}}_{su} \dot{\otimes} \pi^m \bar{\mathbf{X}}_{ut}) \rangle = \langle w, \pi^m \bar{\mathbf{X}}_{su} \dot{\otimes} \pi^m \bar{\mathbf{X}}_{ut} \rangle \\ &= \langle \Delta_{\dot{\otimes}} w, \pi^m \bar{\mathbf{X}}_{su} \otimes \pi^m \bar{\mathbf{X}}_{ut} \rangle = \langle \Delta_{\dot{\otimes}} w, \bar{\mathbf{X}}_{su} \otimes \bar{\mathbf{X}}_{ut} \rangle = (\hat{\mathbf{X}}_{su} \otimes \hat{\mathbf{X}}_{ut}) \Delta_{\dot{\otimes}} w. \end{aligned}$$

Finally, the fact that the analytic condition of Definition 4.2.3 follows from that of Definition 4.2.1 is a direct consequence of Lemma 4.5. Thus, $\check{\mathbf{X}} \in \hat{R}^\gamma([0, T], \mathbb{R}^d)$.

6. I_2^γ is bijective. First of all, if $\mathbf{X}^1, \mathbf{X}^2 \in R^\gamma([0, T], \mathbb{R}^d)$ with $\mathbf{X}^1 \neq \mathbf{X}^2$, then there is $t \in [0, T]$ such that $\mathbf{X}_t^1 \neq \mathbf{X}_t^2$ and thus, there is $w \in \mathcal{W}_d^n$ such that $\langle w, \mathbf{X}_{0t}^1 \rangle \neq \langle w, \mathbf{X}_{0t}^2 \rangle$. Thus, I_2^γ is injective.

Let now $\hat{\mathbf{X}} \in \hat{R}^\gamma([0, T], \mathbb{R}^d)$ be arbitrary. Put $\bar{\mathbf{X}}_{st} := \sum_{w \in \mathcal{W}_d} \hat{\mathbf{X}}(w)w$. Then, we obviously have $\hat{\mathbf{X}}_{st} = \langle \cdot, \bar{\mathbf{X}}_{st} \rangle$ for all $s, t \in [0, T]$. For all $m \in \mathbb{N}$ and $w_1, w_2 \in \mathcal{W}_d^m$ such that $|w_1| + |w_2| \leq m$, we get, again by the duality of \sqcup and Δ_\sqcup ,

$$\begin{aligned} \langle w_1 \otimes w_2, \Delta_\sqcup \bar{\mathbf{X}}_{st} \rangle &= \langle w_1 \otimes w_2, \pi^m \Delta_\sqcup \bar{\mathbf{X}}_{st} \rangle = \langle w_1 \otimes w_2, \Delta_\sqcup \pi^m \bar{\mathbf{X}}_{st} \rangle = \langle w_1 \sqcup w_2, \pi^m \bar{\mathbf{X}}_{st} \rangle \\ &= \langle w_1 \sqcup w_2, \bar{\mathbf{X}}_{st} \rangle = \hat{\mathbf{X}}_{st}(w_1 \sqcup w_2) = \hat{\mathbf{X}}_{st}(w_1) \hat{\mathbf{X}}_{st}(w_2) \\ &= \langle w_1, \bar{\mathbf{X}}_{st} \rangle \langle w_2, \bar{\mathbf{X}}_{st} \rangle = \langle w_1 \otimes w_2, \bar{\mathbf{X}}_{st} \otimes \bar{\mathbf{X}}_{st} \rangle \end{aligned}$$

Since $(\langle \mathcal{W}_d \rangle \otimes \langle \mathcal{W}_d \rangle, \langle \mathcal{W}_d \rangle \otimes \langle \mathcal{W}_d \rangle_\Pi)$ is a pair of dual vector spaces under $\langle \cdot, \cdot \rangle$, we conclude $\Delta_\sqcup \bar{\mathbf{X}}_{st} = \bar{\mathbf{X}}_{st} \otimes \bar{\mathbf{X}}_{st}$ and together with $\langle \mathbf{1}, \bar{\mathbf{X}}_{st} \rangle = \hat{\mathbf{X}}_{st}(\mathbf{1}) = 1$ therefore $\bar{\mathbf{X}}_{st} \in G_d$ for all $s, t \in [0, T]$. Furthermore, for all $m \in \mathbb{N}$, $w \in \mathcal{W}_d^m$ and all $s, u, t \in [0, T]$, we have

$$\begin{aligned} \langle w, \bar{\mathbf{X}}_{st} \rangle &= \hat{\mathbf{X}}_{st}(w) = (\hat{\mathbf{X}}_{su} \otimes \hat{\mathbf{X}}_{ut}) \Delta_{\dot{\otimes}} w = \langle \Delta_{\dot{\otimes}} w, \bar{\mathbf{X}}_{su} \otimes \bar{\mathbf{X}}_{ut} \rangle = \langle \Delta_{\dot{\otimes}} w, \pi^m \bar{\mathbf{X}}_{su} \otimes \pi^m \bar{\mathbf{X}}_{ut} \rangle \\ &= \langle w, \pi^m \bar{\mathbf{X}}_{su} \dot{\otimes}_m \pi^m \bar{\mathbf{X}}_{ut} \rangle = \langle w, \pi^m (\pi^m \bar{\mathbf{X}}_{su} \dot{\otimes} \pi^m \bar{\mathbf{X}}_{ut}) \rangle = \langle w, \pi^m \bar{\mathbf{X}}_{su} \dot{\otimes} \pi^m \bar{\mathbf{X}}_{ut} \rangle \\ &= \langle w, \pi^m (\bar{\mathbf{X}}_{su} \dot{\otimes} \bar{\mathbf{X}}_{ut}) \rangle = \langle w, \bar{\mathbf{X}}_{su} \dot{\otimes}_n \bar{\mathbf{X}}_{ut} \rangle. \end{aligned}$$

Since $(\langle \mathcal{W}_d \rangle, \langle \mathcal{W}_d \rangle_\Pi)$ is a pair of dual vector spaces under $\langle \cdot, \cdot \rangle$, we conclude $\bar{\mathbf{X}}_{st} = \bar{\mathbf{X}}_{su} \dot{\otimes} \bar{\mathbf{X}}_{ut}$. Thus, putting $\bar{\mathbf{X}}_t := \mathbf{X}_{0t}$ for all $t \in [0, T]$ and using $\bar{\mathbf{X}}_{00} = \varepsilon$, we get

$$\mathbf{1} = \bar{\mathbf{X}}_{00} = \bar{\mathbf{X}}_{0t} \dot{\otimes} \bar{\mathbf{X}}_{t0} = \bar{\mathbf{X}}_t \dot{\otimes} \bar{\mathbf{X}}_{t0}$$

and therefore $\bar{\mathbf{X}}_{st} = \bar{\mathbf{X}}_{s0} \dot{\otimes} \bar{\mathbf{X}}_{0t} = \bar{\mathbf{X}}_s^{\dot{\otimes}^{-1}} \dot{\otimes} \bar{\mathbf{X}}_t$. Hence, using Lemma 4.5 and the analytic condition of Definition 4.2.3, we get the analytic condition of Definition 4.2.1 and conclude that indeed $\check{\mathbf{X}} \in \hat{R}^\gamma([0, T], \mathbb{R}^d)$. This means we have shown that I_2^γ is also surjective. \square

Theorem 4.7. (Section 4.4 [Hai14])

Consider maps $\Pi : [0, T] \rightarrow \mathbf{L}(\langle \mathcal{W}_d \rangle, \mathbf{C}([0, T], \mathbb{R}^d))$ and $\Gamma : [0, T]^2 \rightarrow \mathbf{L}(\langle \mathcal{W}_d \rangle, \langle \mathcal{W}_d \rangle)$. They satisfy the conditions

1. $(\Pi_s \mathbf{1})(t) = 1$ and $\Gamma_{st}(x_1 \sqcup x_2) = \Gamma_{st}x_1 \sqcup \Gamma_{st}x_2$ for all $s, t \in [0, T]$, $x_1, x_2 \in \langle \mathcal{W}_d \rangle$,
2. $\Pi_s = \Pi_u \Gamma_{us}$ and $\Gamma_{su} \Gamma_{ut} = \Gamma_{st}$ and $\Delta_{\otimes} \Gamma_{st} = (\Gamma_{st} \otimes \text{id}) \Delta_{\otimes}$ for all $s, u, t \in [0, T]$,
3. $|(\Pi_s w)(t)| \lesssim |t - s|^{\gamma|w|}$ and $\|\Gamma_{st} w\|_m \lesssim |t - s|^{\gamma(|w| - m)}$ for all words w and $m < |w|$, uniformly over all $s, t \in [0, T]$

if and only if they are given by

$$(\Pi_s x)(t) := \hat{\mathbf{X}}_{st}(x), \quad \Gamma_{st} x := (\hat{\mathbf{X}}_{ts} \otimes \text{id}) \Delta_{\otimes} x = \sum_{(x)}^{\otimes} \hat{\mathbf{X}}_{ts}(x_1) x_2$$

for some geometric rough path $\hat{\mathbf{X}} \in \hat{\mathbf{R}}^\gamma([0, T], \mathbb{R}^d)$. In this case, we furthermore have $\Gamma_{st} w - w \in \langle \mathcal{W}_d^{n-1} \rangle$ for all words w with $|w| = n$ and $\Pi_t(x_1 \sqcup x_2) = \Pi_t x_1 \Pi_t x_2$ for all $t \in [0, T]$ and $x_1, x_2 \in \langle \mathcal{W}_d \rangle$.

Proof.

- a) \implies : Let $\hat{\mathbf{X}}_{st} := \varepsilon \Gamma_{ts}$, where ε is the counit of $(\langle \mathcal{W}_d \rangle, \Delta_{\otimes})$. We have

$$(\Pi_t \cdot)(t) = \varepsilon(\cdot)$$

for all $t \in [0, T]$ since $(\Pi_t \mathbf{1})(t) = 1$ and $(\Pi_t w)(t) = 0$ for all words w such that $|w| > 0$. The latter is due to 3. and the continuity of $\Pi_t w$. Hence, using 2. we get

$$\hat{\mathbf{X}}_{st}(x) = \varepsilon \Gamma_{ts} x = (\Pi_t \Gamma_{ts} x)(t) = (\Pi_s x)(t).$$

Due to the counit property (2.5) and 2. it holds that

$$\Gamma_{st} = (\varepsilon \otimes \text{id}) \Delta_{\otimes} \Gamma_{st} = (\varepsilon \otimes \text{id})(\Gamma_{st} \otimes \text{id}) \Delta_{\otimes} = (\varepsilon \Gamma_{st} \otimes \text{id}) \Delta_{\otimes} = (\hat{\mathbf{X}}_{ts} \otimes \text{id}) \Delta_{\otimes}.$$

1. and the fact that the counit is an algebra homomorphism imply

$$\hat{\mathbf{X}}_{st}(w_1 \sqcup w_2) = \varepsilon \Gamma_{ts}(w_1 \sqcup w_2) = \varepsilon(\Gamma_{ts} w_1 \sqcup \Gamma_{ts} w_2) = \varepsilon \Gamma_{ts} w_1 \sqcup \varepsilon \Gamma_{ts} w_2 = \hat{\mathbf{X}}_{st}(w_1) \sqcup \hat{\mathbf{X}}_{st}(w_2),$$

while from 2. we conclude

$$\hat{\mathbf{X}}_{st} = \varepsilon \Gamma_{ts} = \varepsilon \Gamma_{tu} \Gamma_{us} = \hat{\mathbf{X}}_{tu} \Gamma_{us} = \hat{\mathbf{X}}_{ut} (\hat{\mathbf{X}}_{su} \otimes \text{id}) \Delta_{\otimes} = (\hat{\mathbf{X}}_{su} \otimes \hat{\mathbf{X}}_{ut}) \Delta_{\otimes}.$$

Finally, using the fact that $\|\cdot\|_0 = |\varepsilon(\cdot)|$, 3. leads to

$$|\hat{\mathbf{X}}_{st}(w)| = |\varepsilon \Gamma_{ts} w| = \|\Gamma_{ts} w\|_0 \lesssim |t - s|^{\gamma|w|}.$$

Hence, $\hat{\mathbf{X}} \in \hat{\mathbf{R}}^\gamma([0, T], \mathbb{R}^d)$ is a geometric rough path according to Definition 4.2.3.

- b) \longleftarrow : First of all, continuity of $\Pi_s x$ for all $s \in [0, T]$ and all $x \in \mathcal{W}_d$ follows from the fact that for all $u \in [0, T]$, we have

$$\begin{aligned} \lim_{t \rightarrow u} (\Pi_s x)(t) &= \lim_{t \rightarrow u} \hat{\mathbf{X}}_{st}(x) = \sum_{(x)}^{\otimes} \hat{\mathbf{X}}_{su}(x_1) \lim_{t \rightarrow u} \hat{\mathbf{X}}_{ut}(x_2) = \sum_{(x)}^{\otimes} \hat{\mathbf{X}}_{su}(x_1) \varepsilon(x_2) \\ &= \hat{\mathbf{X}}_{su}(x) = (\Pi_s x)(u), \end{aligned}$$

where we used Definition 4.2.3 2. and 3. as well as the counit property (2.5). Due to Definition 4.2.3 1. we have

$$(\Pi_t \mathbf{1})(t) = \hat{\mathbf{X}}_{tt}(\mathbf{1}) = 1.$$

Again using Definition 4.2.3 1. and the fact that Δ_{\otimes} is an algebra homomorphism, we compute

$$\begin{aligned} \Gamma_{st}(x \sqcup y) &= (\hat{\mathbf{X}}_{ts} \otimes \text{id})\Delta_{\otimes}(x \sqcup y) = \sum_{(x)}^{\otimes} \sum_{(y)}^{\otimes} \hat{\mathbf{X}}_{ts}(x_1 \sqcup y_1)(x_2 \sqcup y_2) \\ &= \sum_{(x)}^{\otimes} \sum_{(y)}^{\otimes} \hat{\mathbf{X}}_{ts}(x_1) \hat{\mathbf{X}}_{ts}(y_1)(x_2 \sqcup y_2) = \left(\sum_{(x)}^{\otimes} \hat{\mathbf{X}}_{ts}(x_1)x_2 \right) \sqcup \left(\sum_{(y)}^{\otimes} \hat{\mathbf{X}}_{ts}(y_1)y_2 \right) \\ &= \Gamma_{st}x \sqcup \Gamma_{st}y \end{aligned}$$

Likewise,

$$(\Pi_s(x_1 \sqcup x_2))(t) = \hat{\mathbf{X}}_{st}(x_1 \sqcup x_2) = \hat{\mathbf{X}}_{st}(x_1) \hat{\mathbf{X}}_{st}(x_2) = (\Pi_s x_1)(t) (\Pi_s x_2)(t) = (\Pi_s x_1 \Pi_s x_2)(t).$$

By Definition 4.2.3 2. we conclude

$$(\Pi_u \Gamma_{us} x)(t) = \hat{\mathbf{X}}_{ut}(\Gamma_{us} x) = \hat{\mathbf{X}}_{ut}(\hat{\mathbf{X}}_{su} \otimes \text{id})\Delta_{\otimes} x = (\hat{\mathbf{X}}_{su} \otimes \hat{\mathbf{X}}_{ut})\Delta_{\otimes} x = \hat{\mathbf{X}}_{st}(x) = (\Pi_s x)(t),$$

as well as, using coassociativity,

$$\begin{aligned} \Gamma_{su} \Gamma_{ut} &= (\hat{\mathbf{X}}_{us} \otimes \text{id})\Delta_{\otimes}(\hat{\mathbf{X}}_{tu} \otimes \text{id})\Delta_{\otimes} = (\hat{\mathbf{X}}_{us} \otimes \text{id})(\hat{\mathbf{X}}_{tu} \otimes \Delta_{\otimes})\Delta_{\otimes} \\ &= (\hat{\mathbf{X}}_{tu} \otimes \hat{\mathbf{X}}_{us} \otimes \text{id})(\text{id} \otimes \Delta_{\otimes})\Delta_{\otimes} = (\hat{\mathbf{X}}_{tu} \otimes \hat{\mathbf{X}}_{us} \otimes \text{id})(\Delta_{\otimes} \otimes \text{id})\Delta_{\otimes} \\ &= (\hat{\mathbf{X}}_{ts} \otimes \text{id})\Delta_{\otimes} = \Gamma_{st} \end{aligned}$$

and similarly

$$\begin{aligned} \Delta_{\otimes} \Gamma_{st} &= \Delta_{\otimes}(\hat{\mathbf{X}}_{ts} \otimes \text{id})\Delta_{\otimes} = (\hat{\mathbf{X}}_{ts} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta_{\otimes})\Delta_{\otimes} = (\hat{\mathbf{X}}_{ts} \otimes \text{id} \otimes \text{id})(\Delta_{\otimes} \otimes \text{id})\Delta_{\otimes} \\ &= (\Gamma_{st} \otimes \text{id})\Delta_{\otimes}. \end{aligned}$$

Since for every word w , there are words $w_1^i, w_2^i, |w_1^i| = |w_2^i| = i$ such that $\Delta_{\otimes} w = \sum_{i=0}^{|w|} w_1^{|w|-i} \otimes w_2^i$, we have

$$\|\Gamma_{st} w\|_m \leq \sum_{i=0}^{|w|} |\hat{\mathbf{X}}_{ts}(w_1^{|w|-i})| \|w_2^i\|_m = |\hat{\mathbf{X}}_{ts}(w_1^{|w|-m})| \lesssim |t-s|^{\gamma(|w|-m)} \quad (4.3)$$

due to Definition 4.2.3 3., which also directly implies

$$|(\Pi_s(w))(t)| = |\hat{\mathbf{X}}_{st}(w)| \lesssim |t-s|^{\gamma|w|}$$

for all words w . Finally,

$$\begin{aligned} \Gamma_{st} w &= \sum_{(w)}^{\otimes} \hat{\mathbf{X}}_{ts}(w_1)w_2 = \hat{\mathbf{X}}_{ts}(\mathbf{1})w + \hat{\mathbf{X}}_{ts}(w)\mathbf{1} + \sum_{(w)}^{\otimes} \hat{\mathbf{X}}_{ts}(w')w'' \\ &= w + \hat{\mathbf{X}}_{ts}(w)\mathbf{1} + \sum_{(w)}^{\otimes} \hat{\mathbf{X}}_{ts}(w')w'', \end{aligned}$$

thus $\Gamma_{st} w - w \in \langle \mathcal{W}_d^{n-1} \rangle$ for all words w with $|w| = n$.

□

4.2 Branched rough paths

Put $\mathcal{F}_d := \mathcal{F}_{\{1, \dots, d\}}$ and

$$\mathcal{G}_d := \text{GrLk}(\langle \mathcal{F}_d \rangle_{\Pi}, \Delta_{\circ}) = \{g \in \langle \mathcal{F}_d \rangle_{\Pi} \mid \Delta_{\circ} g = g \otimes g\} = \exp_{\star}(\langle \mathcal{F}_d \rangle_{\Pi}),$$

where

$$\langle \mathcal{F}_d \rangle_{\Pi} = \text{Prim}(\langle \mathcal{F}_d \rangle_{\Pi}, \Delta_{\circ}) = \{l \in \langle \mathcal{F}_d \rangle_{\Pi} \mid \Delta_{\circ} l = \mathbf{1} \otimes l + l \otimes \mathbf{1}\},$$

as well as

$$\mathcal{G}_d^n := \text{GrLk}^n(\langle \mathcal{F}_d^n \rangle, \Delta_{\circ}) = \{g \in \langle \mathcal{F}_d^n \rangle \mid \Delta_{\circ} g = \pi^n(g \otimes g)\} = \pi^n \mathcal{G}_d = \exp_{\star_n}(\langle \mathcal{F}_d^n \rangle),$$

where

$$\langle \mathcal{F}_d^n \rangle = \text{Prim}(\langle \mathcal{F}_d^n \rangle, \Delta_{\circ}) = \{l \in \langle \mathcal{F}_d^n \rangle \mid \Delta_{\circ} l = \mathbf{1} \otimes l + l \otimes \mathbf{1}\}.$$

Define $\|\cdot\|_{\mathcal{G}_d^n} : \mathcal{G}_d^n \rightarrow [0, \infty)$ by (Remark 2.15. [HK14])

$$\left\| \exp_{\star_n} \left(\sum_{\tau \in \mathcal{F}_d^n} b_{\tau} \tau \right) \right\|_{\mathcal{G}_d^n} := \sum_{\tau \in \mathcal{F}_d^n} |b_{\tau}|^{1/|\tau|}.$$

Furthermore, put $\|x\| := \sqrt{\langle x, x \rangle}$ for all $x \in \langle \mathcal{F}_d \rangle$ and $\|x\|_n := \|\pi_n x\|$ for all $x \in \langle \mathcal{F}_d \rangle_{\Pi}$.

Definition 4.8. (Definition 2.13., Remark 2.15. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder branched rough path is a map $\mathbf{X} : [0, T] \rightarrow \mathcal{G}_d^n$ with $\mathbf{X}_0 := \mathbf{1}$ such that

$$\sup_{s < t} \frac{\|\mathbf{X}_{st}\|_{\mathcal{G}_d^n}}{|t - s|^{\gamma}} < \infty,$$

where $\mathbf{X}_{st} := \mathbf{X}_s^{\star_n^{-1}} \star_n \mathbf{X}_t = \mathcal{S}_{\star}^{\circ} \mathbf{X}_s \star_n \mathbf{X}_t$ and \mathbf{n} is the integer part of $\frac{1}{\gamma}$. The set of all such maps $\mathbf{X} : [0, T] \rightarrow \mathcal{G}_d^n$ is denoted by $\mathcal{B}^{\gamma}([0, T], \mathbb{R}^d)$.

Again, we present three more equivalent definitions.

Definition 4.8.1. (Based on Definition 2.13., Remark 2.15. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder branched rough path is a map $\bar{\mathbf{X}} : [0, T] \rightarrow \mathcal{G}_d$ with $\bar{\mathbf{X}}_0 := \mathbf{1}$ such that

$$\sup_{s < t} \frac{\|\pi^n \bar{\mathbf{X}}_{st}\|_{\mathcal{G}_d^n}}{|t - s|^{\gamma}} < \infty \quad \forall n \in \mathbb{N},$$

where $\bar{\mathbf{X}}_{st} := \bar{\mathbf{X}}_s^{\star^{-1}} \star \bar{\mathbf{X}}_t = \mathcal{S}_{\star}^{\circ} \bar{\mathbf{X}}_s \star \bar{\mathbf{X}}_t$. The set of all such maps $\bar{\mathbf{X}} : [0, T] \rightarrow \mathcal{G}_d$ is denoted by $\bar{\mathcal{B}}^{\gamma}([0, T], \mathbb{R}^d)$.

Definition 4.8.2. (Definition 1.6. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder branched rough path is a map $\check{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{F}_d^n \rangle^*$ such that

1. $\check{\mathbf{X}}_{st}(\mathbf{1}) = \mathbf{1}$ and $\check{\mathbf{X}}_{st}(\zeta_1 \zeta_2) = \check{\mathbf{X}}_{st}(\zeta_1) \check{\mathbf{X}}_{st}(\zeta_2)$ for all forests $\zeta_1, \zeta_2 \in \mathcal{F}_d^n$ such that $|\zeta_1| + |\zeta_2| \leq \mathbf{n}$,
2. $\check{\mathbf{X}}_{tt} = \varepsilon_{\mathbf{n}}$ and $\check{\mathbf{X}}_{st}(\zeta) = (\check{\mathbf{X}}_{su} \otimes \check{\mathbf{X}}_{ut}) \Delta_{\star} \zeta = \sum_{(\zeta)}^{\star} \check{\mathbf{X}}_{su}(\zeta^1) \check{\mathbf{X}}_{ut}(\zeta^2)$ for all forests $\zeta \in \mathcal{F}_d^n$,
3. $\sup_{s \neq t} \frac{|\check{\mathbf{X}}_{st}(\zeta)|}{|t - s|^{\gamma|\zeta|}} < \infty$ for all forests $\zeta \in \mathcal{F}_d^n$,

where \mathbf{n} is the integer part of $\frac{1}{\gamma}$. Put $\check{\mathbf{X}}_t := \check{\mathbf{X}}_{0t}$. The set of all such maps $\check{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{F}_d^{\mathbf{n}} \rangle^*$ is denoted by $\check{\mathcal{H}}^\gamma([0, T], \mathbb{R}^d)$.

Definition 4.8.3. (Based on Definition 1.6. [HK14]) Let $\gamma \in (0, 1)$. A d -dimensional γ -Hölder branched rough path is a map $\hat{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{F}_d \rangle^*$ such that

1. $\hat{\mathbf{X}}_{st}(\mathbf{1}) = 1$ and $\hat{\mathbf{X}}_{st}(\zeta_1 \zeta_2) = \hat{\mathbf{X}}_{st}(\zeta_1) \hat{\mathbf{X}}_{st}(\zeta_2)$ for all forests $\zeta_1, \zeta_2 \in \mathcal{F}_d$,
2. $\hat{\mathbf{X}}_{tt} = \varepsilon$ and $\hat{\mathbf{X}}_{st}(\zeta) = (\hat{\mathbf{X}}_{su} \otimes \hat{\mathbf{X}}_{ut}) \Delta_* \zeta = \sum_{(\zeta)}^* \hat{\mathbf{X}}_{su}(\zeta^1) \hat{\mathbf{X}}_{ut}(\zeta^2)$ for all forests $\zeta \in \mathcal{F}_d$,
3. $\sup_{s \neq t} \frac{|\hat{\mathbf{X}}_{st}(\zeta)|}{|t-s|^{\gamma|\zeta|}} < \infty$ for all forests $\zeta \in \mathcal{F}_d$.

Put $\hat{\mathbf{X}}_t := \hat{\mathbf{X}}_{0t}$. The set of all such maps $\hat{\mathbf{X}} : [0, T]^2 \rightarrow \langle \mathcal{F}_d \rangle^*$ is denoted by $\hat{\mathcal{H}}^\gamma([0, T], \mathbb{R}^d)$.

Lemma 4.9. (Based on Remark 2.15. [HK14])

1. For all $k \in \mathbb{N}$, there is a constant $C_k > 0$ such that

$$|\langle \zeta, g \rangle| \leq C_{|w|} \|g\|_{\mathcal{G}_d^n}^{|\zeta|} \quad \forall n \in \mathbb{N}, g \in \mathcal{G}_d^n, \zeta \in \mathcal{F}_d^n \setminus \{\mathbf{1}\}.$$

2. For all $n \in \mathbb{N}$, there is a constant $C'_n > 0$ such that

$$\|g\|_{\mathcal{G}_d^n} \leq C'_n \sup_{\zeta \in \mathcal{F}_d^n \setminus \{\mathbf{1}\}} |\langle \zeta, g \rangle|^{1/|\zeta|} \quad \forall g \in \mathcal{G}_d^n.$$

Proof. (Based on the proof of Proposition 4 [LV07])

1. Let $n \in \mathbb{N}_0$ and $x = \sum_{\tau \in \mathcal{F}_d^n} b_\tau \tau \in \langle \mathcal{F}_d^n \rangle$ be arbitrary. For every $m, k \in \mathbb{N}$ with $m \leq n$, we have

$$\begin{aligned} \|x^{\star_n m}\|_k &= \left\| \left(\sum_{\tau \in \mathcal{F}_d^n \setminus \{\mathbf{1}\}} b_\tau \tau \right)^m \right\|_k \\ &= \left\| \sum_{\substack{\tau_1, \dots, \tau_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}: \\ |\tau_1| + \dots + |\tau_m| = k}} b_{\tau_1} \cdots b_{\tau_m} \tau_1 \star_n \cdots \star_n \tau_m \right\|_k \\ &\leq \underbrace{\sup_{\substack{\tau_1, \dots, \tau_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}: \\ |\tau_1| + \dots + |\tau_m| = k}} \|\tau_1 \star_n \cdots \star_n \tau_m\|_k}_{:= D_{m,k}} \sum_{\substack{\tau_1, \dots, \tau_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}: \\ |\tau_1| + \dots + |\tau_m| = k}} \prod_{i=1}^m |b_{\tau_i}| \\ &\leq D_{m,k} N_{m,k} \sup_{\substack{\tau_1, \dots, \tau_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}: \\ |\tau_1| + \dots + |\tau_m| = k}} \prod_{i=1}^m |b_{\tau_i}| \\ &\leq D_{m,k} N_{m,k} \sup_{\substack{l_1, \dots, l_m \in \mathbb{N}: \\ l_1 + \dots + l_m = k}} \prod_{i=1}^m \|\exp_{\star_n}(x)\|_{\mathcal{G}_d^n}^{l_i} = D_{m,k} N_{m,k} \|\exp_{\star_n}(x)\|_{\mathcal{G}_d^n}^k, \end{aligned}$$

where $N_{m,k} := |\{\tau_1, \dots, \tau_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\} \mid |\tau_1| + \dots + |\tau_m| = k\}| < \infty$. Thus, for any $\zeta \in \mathcal{F}_d^n$, we get

$$\begin{aligned} |\langle \zeta, \exp_{\star_n}(x) \rangle| &= \left| \sum_{m=0}^n \frac{1}{m!} \langle \zeta, x^{\star_n m} \rangle \right| = \left| \sum_{m=0}^{|\zeta|} \frac{1}{m!} \langle \zeta, \pi_{|\zeta|} x^{\star_n m} \rangle \right| \\ &\leq \sum_{m=0}^{|\zeta|} \frac{1}{m!} \|x^{\star_n m}\|_{|\zeta|} \leq \underbrace{\sum_{m=0}^{|\zeta|} \frac{D_{m,|\zeta|} N_{m,|\zeta|}}{m!}}_{:=C_{|\zeta|}} \|\exp_{\star_n}(x)\|_{\mathcal{G}_d^n}^{|\zeta|}. \end{aligned}$$

2. Let $n \in \mathbb{N}_0$ and $u \in \mathfrak{g}(\mathcal{W}_d^n)$ be arbitrary. For every $m, k \in \mathbb{N}$ with $m \leq n$, we have, with

$$\tilde{D}_{m,k} := \sup_{\substack{\zeta_1, \dots, \zeta_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}: \\ |\zeta_1| + \dots + |\zeta_m| = k}} \|\zeta_1 \star_n \dots \star_n \zeta_m\|_k,$$

$\tilde{N}_{m,k} := |\{\zeta_1, \dots, \zeta_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\} \mid |\zeta_1| + \dots + |\zeta_m| = k\}| < \infty$ and similar first steps as above,

$$\begin{aligned} \|u^{\star_n m}\|_k^{1/k} &\leq \underbrace{\tilde{D}_{m,k}^{1/k} \tilde{N}_{m,k}^{1/k}}_{:=R_{m,k}} \sup_{\substack{\zeta_1, \dots, \zeta_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}: \\ |\zeta_1| + \dots + |\zeta_m| = k}} \prod_{i=1}^m |\zeta_i, u|^{1/k} \\ &\leq R_{m,k}^{1/k} \sup_{\substack{\zeta_1, \dots, \zeta_m \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}: \\ |\zeta_1| + \dots + |\zeta_m| = k}} \prod_{i=1}^m \left(\sup_{\zeta \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}} |\langle \zeta, u \rangle|^{1/|\zeta|} \right)^{|\zeta_i|/k} = R_{m,k}^{1/k} \sup_{\zeta \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}} |\langle \zeta, u \rangle|^{1/|\zeta|}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathbf{1} + u\|_{\mathcal{G}_d^n} &= \sum_{k=1}^n \|\log_{\star_n}(\mathbf{1} + u)\|_k^{1/k} = \sum_{k=1}^n \left\| \sum_{m=1}^k \frac{(-1)^{m-1} u^{\star_n m}}{m} \right\|_k^{1/k} \\ &\leq \sum_{k=1}^n \left(\sum_{m=1}^k \frac{1}{m} \|u^{\star_n m}\|_k \right)^{1/k} \stackrel{(\diamond)}{\leq} \sum_{k=1}^n \sum_{m=1}^k \frac{1}{\sqrt[m]{m}} \|u^{\star_n m}\|_k^{1/k} \\ &\leq \underbrace{\sum_{k=1}^n \sum_{m=1}^k \sqrt[k]{\frac{R_{m,k}}{m}}}_{:=C'_n} \sup_{\zeta \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}} |\langle \zeta, u \rangle|^{1/|\zeta|} = C'_n \sup_{\zeta \in \mathcal{F}_d^k \setminus \{\mathbf{1}\}} |\langle \zeta, \mathbf{1} + u \rangle|^{1/|\zeta|}, \end{aligned}$$

where at (\diamond) , we iteratively used the simple fact that $(a+b)^{1/k} \leq a^{1/k} + b^{1/k}$ for all $a, b > 0$, $k \in \mathbb{N}$. □

For each $\gamma \in (0, 1)$, let $q_\gamma : \langle \mathcal{F}_d \rangle \rightarrow \mathbb{R}$ be the linear map recursively generated by $q_\gamma(\zeta) := 1$ for all $\zeta \in \mathcal{F}_d$ such that $|\zeta| \leq 1/\gamma$ and

$$q_\gamma(\zeta) := \frac{1}{2^{|\zeta|} - 2} \sum_{(\zeta)}^* q_\gamma(\zeta') q_\gamma(\zeta''), \quad q_\gamma(\zeta_1 \zeta_2) := q_\gamma(\zeta_1) q_\gamma(\zeta_2)$$

for all $\zeta, \zeta_1, \zeta_2 \in \mathcal{F}_d$ such that $|\zeta| > 1/\gamma$ and $|\zeta_1 \zeta_2| > 1/\gamma$ (Equation (34) [Gub10]).

Theorem 4.10. Gubinelli's Extension Theorem (Theorem 7.3. [Gub10]) *For $\gamma \in (0, 1)$, let $\check{\mathbf{X}} \in \check{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ and $B \in [0, 1], A > 0$ such that*

$$\sup_{s < t} \frac{|\check{\mathbf{X}}_{st}(\tau)|}{|t - s|^{\gamma|\tau|}} \leq BA^{|\tau|} q_\gamma(\tau) \quad \forall \tau \in \mathcal{T}_d^{\mathbf{n}}.$$

Then, there is an extension $\hat{\mathbf{X}} \in \hat{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ such that $\hat{\mathbf{X}}_{st} \downarrow_{\langle \mathcal{F}_d^{\mathbf{n}} \rangle} = \check{\mathbf{X}}_{st}$ for all $s, t \in [0, T]$, where \mathbf{n} is the integer part of γ , and

$$\sup_{s < t} \frac{|\hat{\mathbf{X}}_{st}(\tau)|}{|t - s|^{\gamma|\tau|}} \leq BA^{|\tau|} q_\gamma(\tau) \quad \forall \tau \in \mathcal{T}_d.$$

Lemma 4.11. *If for an $\mathbf{X} \in \mathcal{R}^\gamma([0, T], \mathbb{R}^d)$ there exists an extension $\bar{\mathbf{X}} \in \bar{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ such that $\pi^{\mathbf{n}} \bar{\mathbf{X}} = \mathbf{X}$, then this extension is unique.*

Proof. Assume for $\bar{\mathbf{X}}^1, \bar{\mathbf{X}}^2 \in \bar{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$, we have $\pi^m \bar{\mathbf{X}}^1 = \mathbf{X} = \pi^m \bar{\mathbf{X}}^2$ for some $m \geq \mathbf{n}$. Put $A_t := \log_{\star_{m+1}}(\pi^{m+1} \bar{\mathbf{X}}_t^1)$, $a_t := A_t - \pi^m A_t$, $B_t := \log_{\star_{m+1}}(\pi^{m+1} \bar{\mathbf{X}}_t^2)$ and $b_t := B_t - \pi^m B_t$. Then, since $\pi^m A_t \star_{m+1} a_t = 0$ due to $a_{st} \in \langle \mathcal{F}_{d, m+1} \rangle$ and $\varepsilon A_t = 0$, we have $A_t^{\star_{m+1} k} = (\pi^m A_t)^{\star_{m+1} k}$ for $k > 1$ and thus

$$\pi^{m+1} \bar{\mathbf{X}}_t^a = \exp_{\star_{m+1}}(A_t) = \exp_{\star_{m+1}}(\pi^m A_t) + a_t.$$

Since both $\pi^m A_s$ and a_s are primitive elements, we have $\mathcal{S}_\star^\circ \exp_{\star_{m+1}}(\pi^m A_s) = \exp_{\star_{m+1}}(-\pi^m A_s)$ and $\mathcal{S}_\star^\circ a_s = -a_s$, and thus

$$\begin{aligned} \pi^{m+1} \bar{\mathbf{X}}_{st}^1 &= \pi^{m+1} (\bar{\mathbf{X}}_t \star \mathcal{S}_\star^\circ \bar{\mathbf{X}}_s) = \pi^{m+1} \bar{\mathbf{X}}_t \star_{m+1} \mathcal{S}_\star^\circ \pi^{m+1} \bar{\mathbf{X}}_s \\ &= (\exp_{\star_{m+1}}(\pi^m A_t) + a_t) \star_{m+1} \mathcal{S}(\exp_{\star_{m+1}}(\pi^m A_s) + a_s) \\ &= (\exp_{\star_{m+1}}(\pi^m A_t) + a_t) \star_{m+1} (\exp_{\star_{m+1}}(-\pi^m A_s) - a_s) \\ &= \exp_{\star_{m+1}}(\pi^m A_t) \star_{m+1} \exp_{\star_{m+1}}(-\pi^m A_s) + a_t - a_s \end{aligned}$$

Likewise, $\pi^{m+1} \bar{\mathbf{X}}_{st}^2 = \exp_{\star_{m+1}}(\pi^m B_t) \star_{m+1} \exp_{\star_{m+1}}(\pi^m B_s) + b_t - b_s$. Since

$$\begin{aligned} \pi^m A_t &= \pi^m \log_{\star_{m+1}}(\pi^{m+1} \bar{\mathbf{X}}_t^1) = \log_{\star_m}(\pi^m \bar{\mathbf{X}}_t^1) = \log_{\star_m}(\pi^m \bar{\mathbf{X}}_t^2) = \pi^m \log_{\star_{m+1}}(\pi^{m+1} \bar{\mathbf{X}}_t^2) \\ &= \pi^m B_t, \end{aligned}$$

we have

$$\pi^{m+1} \bar{\mathbf{X}}_{st}^1 - \pi^{m+1} \bar{\mathbf{X}}_{st}^2 = a_t + b_s - a_s - b_t.$$

Using Lemma 4.9, we get from the analytic condition in Definition 4.8.1 that

$$\sup_{s < t} \frac{|\langle \zeta, (a_t - b_t) - (a_s - b_s) \rangle|}{|t - s|^{(m+1)\gamma}} < \infty \quad \forall \zeta \in \mathcal{F}_{d, m+1},$$

thus $a - b$ is constant since $(m+1)\gamma \geq (\mathbf{n}+1)\gamma > 1$, and hence $\pi^{m+1} \bar{\mathbf{X}}_{st}^1 = \pi^{m+1} \bar{\mathbf{X}}_{st}^2$.

By inductive application of the argumentation above, we get $\bar{\mathbf{X}}^1 = \bar{\mathbf{X}}^2$. □

Theorem 4.12. *Let $\gamma \in (0, 1)$ and \mathbf{n} denote the integer part of $\frac{1}{\gamma}$. The maps*

$$\begin{aligned} I_1^\gamma &: \bar{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d) \rightarrow \mathcal{R}^\gamma([0, T], \mathbb{R}^d), \bar{\mathbf{X}} \mapsto \mathbf{X}, \mathbf{X}_t := \pi^{\mathbf{n}} \bar{\mathbf{X}}_t, \\ I_2^\gamma &: \mathcal{R}^\gamma([0, T], \mathbb{R}^d) \rightarrow \check{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d), \mathbf{X} \mapsto \check{\mathbf{X}}, \check{\mathbf{X}}_{st}(y) := \langle y, \mathbf{X}_{st} \rangle, \\ I_3^\gamma &: \bar{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d) \rightarrow \hat{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d), \bar{\mathbf{X}} \mapsto \hat{\mathbf{X}}, \hat{\mathbf{X}}_{st}(y) := \langle y, \bar{\mathbf{X}}_{st} \rangle \end{aligned}$$

are well-defined and bijective.

Proof. The proof that the functions I_2^γ and I_3^γ are well-defined and bijective is completely analogous to steps 3. to 6. of the proof of Theorem 4.6. It remains to look at I_1^γ .

1. $I_1^\gamma : \bar{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d) \rightarrow \mathcal{R}^\gamma([0, T], \mathbb{R}^d)$ is well-defined. For each $\bar{\mathbf{X}} \in \bar{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$, we have that $\mathbf{X} := \pi^n \bar{\mathbf{X}} \in \mathcal{R}^\gamma([0, T], \mathbb{R}^d)$ since $\pi^n \mathcal{G}_d = \mathcal{G}_d^n$ (see Corollary 3.40) and

$$\pi^n(\bar{\mathbf{X}}_{st}) = \pi^n(\mathcal{S}_*^\circ \bar{\mathbf{X}}_s \star \bar{\mathbf{X}}_t) = \pi^n \mathcal{S}_*^\circ \bar{\mathbf{X}}_s \star_n \pi^n \bar{\mathbf{X}}_t = \mathcal{S}_*^\circ \pi^n \bar{\mathbf{X}}_s \star_n \pi^n \bar{\mathbf{X}}_t = \mathcal{S}_*^\circ \mathbf{X}_s \star_n \mathbf{X}_t,$$

where the latter together with the analytic condition of Definition 4.8.1 implies the analytic condition of Definition 4.8.

2. I_1^γ is bijective. Injectivity was already proven in Lemma 4.11. To show surjectivity, let $\mathbf{X} \in \mathcal{R}^\gamma([0, T], \mathbb{R}^d)$ be arbitrary. Then, $\check{\mathbf{X}} := I_2^\gamma(\mathbf{X}) \in \check{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ is given by $\check{\mathbf{X}}_{st}(y) = \langle y, \mathbf{X}_{st} \rangle$ for all $y \in \langle \mathcal{F}_d^n \rangle$, $s, t \in [0, T]$. Since \mathcal{F}_d^n is a finite set, for any $B \in (0, 1]$ there is some $A > 0$ such that

$$\sup_{s < t} \frac{|\check{\mathbf{X}}_{st}(\tau)|}{|t - s|^{\gamma|\tau|}} \leq BA^{|\tau|} q_\gamma(\tau) \quad \forall \tau \in \mathcal{F}_d^n.$$

Thus, applying Theorem 4.10, we find that there is an extension $\hat{\mathbf{X}} \in \hat{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ such that $\hat{\mathbf{X}}_{st}|_{\langle \mathcal{F}_d^n \rangle} = \check{\mathbf{X}}_{st}$. By surjectivity of I_3^γ , there finally is $\bar{\mathbf{X}} \in \bar{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ such that $\hat{\mathbf{X}}_{st}(y) = \langle y, \bar{\mathbf{X}}_{st} \rangle$ for all $y \in \langle \mathcal{F}_d \rangle$, $s, t \in [0, T]$. Then, again for all $s, t \in [0, T]$,

$$\begin{aligned} I_3^\gamma(\bar{\mathbf{X}}_{st}) &= \pi^n \bar{\mathbf{X}}_{st} = \pi^n \sum_{\zeta \in \mathcal{F}_d} \langle \zeta, \bar{\mathbf{X}}_{st} \rangle \zeta = \pi^n \sum_{\zeta \in \mathcal{F}_d} \hat{\mathbf{X}}_{st}(\zeta) \zeta = \sum_{\zeta \in \mathcal{F}_d^n} \hat{\mathbf{X}}_{st}(\zeta) \zeta = \sum_{\zeta \in \mathcal{F}_d^n} \check{\mathbf{X}}_{st}(\zeta) \zeta \\ &= \sum_{\zeta \in \mathcal{F}_d^n} \langle \zeta, \mathbf{X}_{st} \rangle \zeta = \mathbf{X}_{st}. \end{aligned}$$

Since $\mathbf{X} \in \mathcal{R}^\gamma([0, T], \mathbb{R}^d)$ was arbitrary, we have proven that I_1^γ is surjective. □

Putting $\|\zeta\| := \sqrt{\langle \zeta, \zeta \rangle}$ and $\|\zeta\|_m := \|\pi_m \zeta\|$, we have the following.

Theorem 4.13. (Based on Section 4.4 [Hai14])

Consider maps $\Pi : [0, T] \rightarrow \mathbb{L}(\langle \mathcal{F}_d \rangle, \mathbb{C}([0, T], \mathbb{R}^d))$ and $\Gamma : [0, T]^2 \rightarrow \mathbb{L}(\langle \mathcal{F}_d \rangle, \langle \mathcal{F}_d \rangle)$. They satisfy the conditions

1. $(\Pi_s(\mathbf{1}))(t) = 1$ and $\Gamma_{st}(x_1 \odot x_2) = \Gamma_{st} x_1 \odot \Gamma_{st} x_2$ for all $s, t \in [0, T]$, $x_1, x_2 \in \langle \mathcal{F}_d \rangle$,
2. $\Pi_s = \Pi_u \Gamma_{us}$ and $\Gamma_{su} \Gamma_{ut} = \Gamma_{st}$ and $\Delta_* \Gamma_{st} = (\Gamma_{st} \otimes \text{id}) \Delta_*$ for all $s, u, t \in [0, T]$,
3. $|(\Pi_s(\zeta))(t)| \lesssim |t - s|^{\gamma|\zeta|}$ and $\|\Gamma_{st} \zeta\|_m \lesssim |t - s|^{\gamma(|\zeta| - m)}$ for all forests ζ and $m < |\zeta|$, uniformly over all $s, t \in [0, T]$

if and only if they are given by

$$(\Pi_s(x))(t) := \hat{\mathbf{X}}_{st}(x), \quad \Gamma_{st} x := (\hat{\mathbf{X}}_{ts} \otimes \text{id}) \Delta_* x = \sum_{(x)}^* \hat{\mathbf{X}}_{ts}(x_1) x_2$$

for some branched rough path $\hat{\mathbf{X}} \in \hat{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$. In this case, we furthermore have $\Gamma_{st} \zeta - \zeta \in \langle \mathcal{F}_d^{n-1} \rangle$ for all forests ζ with $|\zeta| = n$ and $\Pi_t(x_1 \odot x_2) = \Pi_t x_1 \Pi_t x_2$ for all $t \in [0, T]$ and $x_1, x_2 \in \langle \mathcal{F}_d \rangle$.

Proof. Analogous to the proof of Theorem 4.7. Just replace $(\langle \mathcal{W}_d \rangle, \sqcup, \Delta_\otimes)$ with $(\langle \mathcal{F}_d \rangle, \odot, \Delta_*)$ everywhere, except that instead of (4.3) we have now

$$\begin{aligned} \|\Gamma_{st}\zeta\|_m &\leq \sum_{(C,T) \in \text{Cuts}(\zeta)} c_\zeta(C,T) |\hat{\mathbf{X}}_{ts}(C)| \|T\|_m = \sum_{\substack{(C,T) \in \text{Cuts}(\zeta): \\ |C|=|\zeta|-m}} c_\zeta(C,T) |\hat{\mathbf{X}}_{ts}(C)| \\ &\lesssim |t-s|^{\gamma(|\zeta|-m)}. \end{aligned}$$

□

It turns out that weakly geometric rough paths can be seen as a special kind of branched rough paths satisfying an additional condition, which is nothing but the integration by parts rule. This is made precise by the following result, which is due to [HK14]. We added a proof for the 'conversely' part which was just stated as an observation there.

Let $\phi : \langle \mathcal{F}_d \rangle \rightarrow \langle \mathcal{W}_d \rangle$ be the Hopf algebra homomorphism and $\hat{\phi} : \langle \mathcal{W}_d \rangle \rightarrow \langle \mathcal{F}_d \rangle$ be the coalgebra monomorphism introduced in Theorem 3.55.

Theorem 4.14. (Section 4.1, Proposition 4.6. [HK14]) *For every weakly geometric rough path $\check{\mathbf{X}}^g \in \mathcal{R}_2^\gamma([0, T], \mathbb{R}^d)$, there is a branched rough path $\check{\mathbf{X}}^b \in \mathcal{R}_2^\gamma([0, T], \mathbb{R}^d)$ given by*

$$\check{\mathbf{X}}_{st}^b(x) := \check{\mathbf{X}}_{st}^g(\phi(x)).$$

$\check{\mathbf{X}}^g$ can be recovered from $\check{\mathbf{X}}^b$ by

$$\check{\mathbf{X}}_{st}^g(x) = \check{\mathbf{X}}_{st}^b(\hat{\phi}(x)).$$

Conversely, for every branched rough path $\check{\mathbf{X}}^{b'} \in \mathcal{R}_2^\gamma([0, T], \mathbb{R}^d)$ with the additional property that

$$\check{\mathbf{X}}_{st}^{b'}(x) = 0 \quad \forall x \in \langle \mathcal{F}_d^{\mathbf{n}} \rangle \cap \ker \phi, \quad s, t \in [0, T], \quad (4.4)$$

there is a weakly geometric rough path $\check{\mathbf{X}}^{g'} \in \mathcal{R}_2^\gamma([0, T], \mathbb{R}^d)$ given by

$$\check{\mathbf{X}}_{st}^{g'}(x) := \check{\mathbf{X}}_{st}^{b'}(\hat{\phi}(x)).$$

$\check{\mathbf{X}}^{b'}$ can be recovered from $\check{\mathbf{X}}^{g'}$ by

$$\check{\mathbf{X}}_{st}^{b'}(x) = \check{\mathbf{X}}_{st}^{g'}(\phi(x)).$$

Proof. Again, put $\mathbf{n} := \lfloor \frac{1}{\gamma} \rfloor$.

1. Let $\check{\mathbf{X}}^g \in \mathcal{R}_2^\gamma([0, T], \mathbb{R}^d)$ and $\check{\mathbf{X}}^b$ defined as above. The algebraic conditions 1. and 2. of Definition 4.8.2 follow then directly from the algebraic conditions 1. and 2. of Definition 4.2.2 due to the fact that ϕ is a bialgebra homomorphism which preserves the grading. We also have

$$\sup_{s < t} \frac{|\check{\mathbf{X}}_{st}^b(\zeta)|}{|t-s|^{\gamma|\zeta|}} = \sup_{s < t} \frac{|\check{\mathbf{X}}_{st}^g(\phi(\zeta))|}{|t-s|^{\gamma|\zeta|}} < \infty \quad \forall \zeta \in \langle \mathcal{F}_d^{\mathbf{n}} \rangle$$

due to condition 3. of Definition 4.2.2 since $\phi(\zeta) \in \langle \mathcal{W}_{d,|\zeta|} \rangle$, and thus we indeed have $\check{\mathbf{X}}^b \in \mathcal{R}_2^\gamma([0, T], \mathbb{R}^d)$. The recovery property is clear from the fact that $\phi\hat{\phi} = \text{id}$.

2. Let $\check{\mathbf{X}}^{b'} \in \mathcal{R}_2^\gamma([0, T], \mathbb{R}^d)$ be such that it satisfies the additional property and let $\check{\mathbf{X}}^{g'}$ be defined as above. We have that $\hat{\phi}(w_1) \odot \hat{\phi}(w_2) - \hat{\phi}(w_1 \sqcup w_2) \in \ker \phi$ for all $w_1, w_2 \in \mathcal{W}_d$, since

$$\begin{aligned} \phi(\hat{\phi}(w_1) \odot \hat{\phi}(w_2) - \hat{\phi}(w_1 \sqcup w_2)) &= \phi(\hat{\phi}(w_1) \odot \hat{\phi}(w_2)) - \phi\hat{\phi}(w_1 \sqcup w_2) \\ &= \phi\hat{\phi}(w_1) \sqcup \phi\hat{\phi}(w_2) - w_1 \sqcup w_2 = 0. \end{aligned}$$

Hence, using property 1. of Definition 4.8.2, we get for all $w_1, w_2 \in \mathcal{W}_d$ with $|w_1| + |w_2| \leq \mathbf{n}$ that

$$\begin{aligned} \check{\mathbf{X}}^{\mathbf{g}'}(w_1 \sqcup w_2) &= \check{\mathbf{X}}^{\mathbf{b}'}(\hat{\phi}(w_1 \sqcup w_2)) = \check{\mathbf{X}}^{\mathbf{b}'}(\hat{\phi}(w_1) \odot \hat{\phi}(w_2)) = \check{\mathbf{X}}^{\mathbf{b}'}(\hat{\phi}(w_1))\check{\mathbf{X}}^{\mathbf{b}'}(\hat{\phi}(w_2)) \\ &= \check{\mathbf{X}}^{\mathbf{g}'}(w_1)\check{\mathbf{X}}^{\mathbf{g}'}(w_2). \end{aligned}$$

Since $\hat{\phi}$ is a coalgebra homomorphism, condition 2. of Definition 4.2.2 for $\check{\mathbf{X}}^{\mathbf{g}'}$ follows directly from condition 2. of Definition 4.8.2. Furthermore,

$$\sup_{s < t} \frac{|\check{\mathbf{X}}_{st}^{\mathbf{g}'}(w)|}{|t - s|^{\gamma|w|}} = \sup_{s < t} \frac{|\check{\mathbf{X}}_{st}^{\mathbf{b}'}(\hat{\phi}(w))|}{|t - s|^{\gamma|\hat{\phi}(w)|}} < \infty \quad \forall w \in \langle \mathcal{W}_d^{\mathbf{n}} \rangle$$

due to condition 3. of Definition 4.8.2, and thus we indeed have $\check{\mathbf{X}}^{\mathbf{g}'} \in \mathbf{R}_2^\gamma([0, T], \mathbb{R}^d)$.

We also have that $\hat{\phi}\phi(x) - x \in \ker \phi$ for all $x \in \langle \mathcal{F}_d \rangle$, since

$$\phi(\hat{\phi}\phi(x) - x) = \phi\hat{\phi}\phi(x) - \phi(x) = \phi(x) - \phi(x) = 0.$$

Thus,

$$\check{\mathbf{X}}^{\mathbf{b}'}(x) = \check{\mathbf{X}}^{\mathbf{b}'}(\hat{\phi}\phi(x)) = \check{\mathbf{X}}^{\mathbf{g}'}(\phi(x)) \quad \forall x \in \langle \mathcal{F}_d^{\mathbf{n}} \rangle.$$

□

Chapter 5

Regularity structures

5.1 General concept

Definition 5.1. (Definition 3.1 [Hai15], Definition 2.1 [Hai14], Definition 13.1 [FH14])

Let

1. $A \subseteq \mathbb{R}$ be bounded from below and locally finite, i.e. $A \cap (-\infty, r)$ finite for all $r \in \mathbb{R}$,
2. $(T_\alpha)_{\alpha \in A}$ be a family of non-trivial (i.e. $\dim T_\alpha \neq 0 \ \forall \alpha \in A$) Banach spaces over \mathbb{R} with norms $(\|\cdot\|_\alpha)_{\alpha \in A}$ and let $T := \bigoplus_{\alpha \in A} T_\beta$ be endowed with the norm $\|\cdot\| := \sum_{\alpha \in A} \|\pi_\alpha \cdot\|_\alpha$, where π_α is the canonical projection on T_α ,
3. $G \subseteq \mathcal{L}(T, T)$ be a group of invertible continuous linear operators with composition law, i.e. $\text{id} \in G$, $\Gamma_1 \Gamma_2 \in G$ and $\Gamma^{-1} \in G$ for all $\Gamma_1, \Gamma_2, \Gamma \in G$, with the property that

$$\Gamma x - x \in T_{<\alpha} := \bigoplus_{\beta \in A: \beta < \alpha} T_\beta \quad \forall \Gamma \in G, \alpha \in A, x \in T_\alpha \quad (5.1)$$

Then, $\mathcal{T} := (A, T, G)$ is called a *regularity structure* with *index set* A , *model space* T and *structure group* G .

We say that a regularity structure $\tilde{\mathcal{T}} = (\bar{A}, \bar{T}, \bar{G})$ is a *sub regularity structure* of a regularity structure $\mathcal{T} = (A, T, G)$ if $\bar{A} \subseteq A$, $\bar{G} \subseteq G$ and \bar{T} is a closed subspace of T such that \bar{T}_α is a sub Banach space of T_α for all $\alpha \in \bar{A}$ (loosely based on Section 2.1 [Hai14]). We put $T_{\geq \alpha} := \bigoplus_{\beta \in A: \beta \geq \alpha} T_\beta$ and shortly write $\|\cdot\|_\alpha$ also for the seminorms $\|\pi_\alpha \cdot\|_\alpha$ on T .

Remark 5.2. For each A and $(T_\alpha)_\alpha$ fulfilling Properties 1. and 2. there is a maximal structure group given by

$$G^{\max} := \{\Gamma \in \mathcal{L}(T, T) \mid \Gamma x - x \in T_{<\alpha} \ \forall \alpha \in A, x \in T_\alpha\},$$

which contains all possible choices of structure groups. Hence, choosing a structure group for a given model space means choosing certain additional properties that the elements have to fulfill, e.g. multiplicity with respect to a given product on T .

Remark 5.3. All of the examples of regularity structures (A, T, G) that we present later will satisfy the additional property that $0 \in A$ and that there is a unique element $\mathbf{1} \in T_0$ such that $\|\mathbf{1}\|_0 = 1$ and $\Gamma \mathbf{1} = \mathbf{1}$ for all $\Gamma \in G$.

The original definition of a regularity structure in [Hai14] contained an even stronger requirement for the subspace T_0 , namely that $T_0 = \langle \mathbf{1} \rangle$, where $\mathbf{1}$ is such that $\|\mathbf{1}\|_0 = 1$ and $\Gamma \mathbf{1} = \mathbf{1}$ for all $\Gamma \in G$. Definition 5.1 here is exactly the one given in [Hai15].

Definition 5.4. (Definition 2.17 [Hai14]) Let $\mathcal{T} = (A, T, G)$ be a regularity structure and (S, d) be a metric space. A map $\Gamma : S \times S \rightarrow G$ is called a G -bimap¹ over S if

1. $\Gamma_{xx} = \text{id}$ for all $x \in S$,
2. $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ for all $x, y, z \in S$,
3. For every compact subset $\mathfrak{K} \subseteq S$ and every $\beta \in A$, there is a constant $C_{\mathfrak{K}, \beta}$ such that

$$\|\Gamma_{xy}a\|_{\theta} \leq C_{\mathfrak{K}, \beta} \|a\| d(x, y)^{\beta - \theta} \quad \forall x, y \in \mathfrak{K}, a \in T_{\beta}, \theta < \beta.$$

Definition 5.5. For a regularity structure $\mathcal{T} = (A, T, G)$, let V be a subspace of T such that $V_{\alpha} := V \cap T_{\alpha}$ is a (possibly $\{0\}$) closed subspace of T_{α} for all $\alpha \in A$. Then, V is called a *sector* of \mathcal{T} if V is invariant under G , i.e.

$$\Gamma V \subseteq V \quad \forall \Gamma \in G.$$

We call the sector *function-like* if $V_{\alpha} = \{0\}$ for all $\alpha \in A$ with $\alpha < 0$ and $V_0 = \langle \mathbf{1} \rangle$, where $\mathbf{1}$ is some element such that $\Gamma \mathbf{1} = \mathbf{1}$ for all $\Gamma \in G$.

Definition 5.6. (Definition 3.1 [Hai14]) For a given regularity structure (A, T, G) , a G -bimap Γ over (S, d) , $\gamma > \min A$, a closed subset $M \subseteq S$ and a sector V of (A, T, G) , the space of *modelled distributions* $\mathcal{D}_{\Gamma}^{\gamma}(M, V)$ of order γ consists of all functions $f : M \rightarrow V_{< \gamma}$ such that

$$\|f\|_{\mathfrak{K}} := \sup_{x \in \mathfrak{K}} \sup_{\alpha < \gamma} \|f(x)\|_{\alpha} + \sup_{\substack{x, y \in \mathfrak{K}: \\ 0 < d(x, y) \leq 1}} \sup_{\alpha < \gamma} \frac{\|f(x) - \Gamma_{xy}f(y)\|_{\alpha}}{d(x, y)^{\gamma - \alpha}} < \infty$$

for all compact $\mathfrak{K} \subseteq M$. We shortly write $\mathcal{D}_{\Gamma}^{\gamma} := \mathcal{D}_{\Gamma}^{\gamma}(S, T)$.

The following definition is inspired by the definition of automorphisms in Section 2.4 [Hai14].

Definition 5.7. Let $\mathcal{T} = (A, T, G)$ and $\bar{\mathcal{T}} = (\bar{A}, \bar{T}, \bar{G})$ be regularity structures. Furthermore, let $\varphi : T \rightarrow \bar{T}$ be a continuous linear map such that $\varphi T_{\beta} \subseteq \bar{T}_{\beta}^{+}$ for all $\beta \in A$ and $\psi : \bar{G} \rightarrow G$ be a group homomorphism. If

1. $\Gamma \varphi = \varphi \psi(\Gamma) \quad \forall \Gamma \in \bar{G}$,
2. for every G -bimap Γ the function $(x, y) \mapsto \psi(\Gamma_{xy})$ is a \bar{G} -bimap,

then φ is called a model space morphism and ψ is called a structure group morphism. The pair (φ, ψ) is called a regularity structure morphism from $\bar{\mathcal{T}}$ to \mathcal{T} . The morphisms φ , ψ and (φ, ψ) are called homogeneous if $\varphi T_{\beta} \subseteq \bar{T}_{\beta}$.

For $x \in \mathbb{R}^d$ and $\delta > 0$ let $\mathfrak{S}_x^{\delta} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$ be given by

$$(\mathfrak{S}_x^{\delta} u)(y) := \delta^{-d} u(\delta^{-1}(y - x)).$$

Furthermore, for any $r \in \mathbb{N}$, let $\mathcal{B}^r(\mathbb{R}^d)$ denote the set of all $u \in \mathcal{S}(\mathbb{R}^d)$ compactly supported in $B(0, 1)$ such that $\|u\|_{C^r} \leq 1$.

¹This nonstandard terminology just used for the purpose of this thesis.

Definition 5.8. (Definition 2.17 [Hai14]) Let $\mathcal{T} = (A, T, G)$ be a regularity structure, $\Pi : \mathbb{R}^d \rightarrow \mathcal{L}(T, \mathcal{S}'(\mathbb{R}^d))$ be a function and Γ be a G -bimap over \mathbb{R}^d . Furthermore, let r be the smallest integer such that $r > |\min A|$. We call (Π, Γ) a model for \mathcal{T} on \mathbb{R}^d if

1. $\Pi_x \Gamma_{xy} = \Pi_y$ for all $x, y \in \mathbb{R}^d$,
2. For every compact subset $\mathfrak{K} \subseteq \mathbb{R}^d$ and every $\beta \in A$, there is a constant $C_{\mathfrak{K}, \beta}$ such that

$$|(\Pi_x a)(\mathfrak{S}_x^\delta u)| \leq C_{\mathfrak{K}, \beta} \|a\| \delta^\beta \quad \forall x \in \mathfrak{K}, \delta \in (0, 1], a \in T_\beta, u \in \mathcal{B}^r(\mathbb{R}^d).$$

Let $\mathcal{M}(\mathbb{R}^d)$ denote the set of all models for \mathcal{T} on \mathbb{R}^d . If \mathcal{T} satisfies the additional property of Remark 5.3, then let $\mathcal{M}_1(\mathbb{R}^d) \subseteq \mathcal{M}(\mathbb{R}^d)$ denote those models with the additional property that $\Pi_x \mathbf{1}$ is the constant one function for all $x \in \mathbb{R}^d$.

Definition 5.9. (Definition 14.3 [FH14], based on Definition 4.1 and Definition 4.6 [Hai14]) Let V, \bar{V} be two sectors of a regularity structure $\mathcal{T} = (A, T, G)$. A *product* on $V \times \bar{V}$ is a continuous bilinear map $\diamond : V \times \bar{V} \rightarrow T$ such that

1. $V_\alpha \diamond V_\beta \subseteq T_{\alpha+\beta}$,
2. $\Gamma(v \diamond \bar{v}) = \Gamma v \diamond \Gamma \bar{v} \quad \forall \Gamma \in G, v \in V, \bar{v} \in \bar{V}$.

Theorem 5.10. (Section 4.2 and Theorem 4.16 [Hai14], Section 14.2 and Proposition 14.7. [FH14]) *Let V be a function-like sector of a regularity structure $\mathcal{T} = (A, T, G)$ with ϑ being the lowest non-zero homogeneity of V and \diamond a product on $V \times V$ such that $V \diamond V \subseteq V$. Furthermore, let $\gamma > 0$, $h \in C^k(\mathbb{R}, \mathbb{R})$ for some $k \in \mathbb{N}$ with $k \geq \gamma/\vartheta + 1$ and Γ a G -bimap over \mathbb{R}^d . Then, $F \mapsto (h \circ F)_{\gamma}$, where*

$$(h \circ F)_{\gamma}(x) := \sum_{n=0}^{\lfloor \gamma/\vartheta \rfloor} \pi_{<\gamma} \frac{h^{(n)}(F^{\mathbf{1}}(x))}{n!} (F(x) - F^{\mathbf{1}}(x)\mathbf{1})^{\circ n},$$

$v^{\circ 0} := \mathbf{1}$ and $F^{\mathbf{1}}(x)\mathbf{1} := \pi_0 F(x)$, is a continuous map from $\mathcal{D}_{\Gamma}^{\gamma}$ into itself. More precisely, for any compact $\mathfrak{K} \subseteq \mathbb{R}^d$ and any $D > 0$, there is a $C_{\mathfrak{K}, D} > 0$ such that

$$\|(h \circ F)_{\gamma} - (h \circ G)_{\gamma}\|_{\gamma, \mathfrak{K}} \leq C_{\mathfrak{K}, D} \|F - G\|_{\gamma, \mathfrak{K}} \quad \forall F, G \in \mathcal{D}_{\Gamma}^{\gamma} : \|F\|_{\gamma, \mathfrak{K}} + \|G\|_{\gamma, \mathfrak{K}} \leq D.$$

Definition 5.11. (Section 2.2 and Definition 3.7 [Hai14])

1. For $n \in \mathbb{N}_0$ and $\alpha \in (0, 1]$, let $\mathcal{C}^{n+\alpha}(\mathbb{R}^d)$ denote those functions in $C(\mathbb{R}^d, \mathbb{R})$ which are n times differentiable and whose n -th derivative is Hölder continuous with exponent α . We endow $\mathcal{C}^{n+\alpha}(\mathbb{R}^d)$ with the family of seminorms

$$\|f\|_{\mathcal{C}^{n+\alpha}, \mathfrak{K}} := \sup_{x, y \in \mathfrak{K}} \frac{|f^{(n)}(x) - f^{(n)}(y)|}{\|x - y\|^\alpha} + \sum_{k=0}^n \sup_{x \in \mathfrak{K}} |f^{(k)}(x)|$$

indexed by all compact $\mathfrak{K} \subset \mathbb{R}^d$.

2. Put $\mathcal{C}^0(\mathbb{R}^d) := \mathcal{L}_{\text{loc}}^\infty(\mathbb{R}^d, \mathbb{R})$.
3. For $n \in \mathbb{N}$ and $\alpha \in [0, 1)$, let $\mathcal{C}^{\alpha-n}(\mathbb{R}^d)$ denote those distributions $\varrho \in \mathcal{D}'(\mathbb{R}^d)$ such that for each $\mathfrak{K} \subset \mathbb{R}^d$, we have

$$\|\varrho\|_{\mathcal{C}^{\alpha-n}, \mathfrak{K}} := \sup_{u \in \mathcal{B}^n(\mathbb{R}^d)} \sup_{x \in \mathfrak{K}} \sup_{\delta \in (0, 1]} \frac{\varrho(\mathfrak{S}_x^\delta u)}{\delta^{\alpha-n}} < \infty.$$

We endow $\mathcal{C}^{\alpha-n}(\mathbb{R}^d)$ with this family of seminorms.

Theorem 5.12. Reconstruction Theorem (Theorem 3.10 and Corollary [Hai14]) *Let (Π, Γ) be a model over \mathbb{R}^d for a regularity structure $\mathcal{T} = (A, T, G)$ and $\gamma > \alpha := \min A$. Then, there is a continuous linear map $\mathcal{R}^\alpha : \mathcal{D}_\Gamma^\gamma \rightarrow \mathcal{C}^\alpha(\mathbb{R}^d, \mathbb{R})$ with the property that for every compact $\mathfrak{K} \subset \mathbb{R}^d$ there is a constant $C_{\mathfrak{K}} > 0$ such that*

$$|(\mathcal{R}^\alpha F - \Pi_x F(x))(\mathfrak{S}_x^\delta u)| \leq C_{\mathfrak{K}} \|F\|_{\mathfrak{K}, \delta} \delta^\gamma \quad \forall x \in \mathfrak{K}, \delta \in (0, 1], F \in \mathcal{D}_\Gamma^\gamma, u \in \mathcal{B}^r(\mathbb{R}^d), \quad (5.2)$$

where $\mathfrak{K}' := \bigcup_{x \in \mathfrak{K}} \bar{B}(x, 1)$. Such a map is called a reconstruction map. We furthermore have that for every compact $\mathfrak{K} \subset \mathbb{R}^d$ there is a $C'_{\mathfrak{K}}$ such that

$$\|\mathcal{R}^\alpha F\|_{\mathcal{C}^\alpha, \mathfrak{K}} \leq C'_{\mathfrak{K}} \|F\|_{\gamma, \mathfrak{K}'}$$

If $\gamma > 0$, then the reconstruction map is unique and for each $F \in \mathcal{D}_\Gamma^\gamma$, the distribution $\mathcal{R}F$ is uniquely characterized by equation (5.2).

Theorem 5.13. (Special case of Proposition 3.28 [Hai14]) *Let V be a function-like sector of a regularity structure $\mathcal{T} = (A, T, G)$ with ϑ being the lowest non-zero homogeneity of V . Let (Π, Γ) be a model for \mathcal{T} on \mathbb{R}^d such that $\Pi_x \mathbf{1}$ is the constant one function for all $x \in \mathbb{R}^d$. Then, for $\gamma > \vartheta$, \mathcal{R}^γ maps $\mathcal{D}_\Gamma^\gamma(V)$ into $\mathcal{C}^\vartheta(\mathbb{R}^d)$ and we have*

$$(\mathcal{R}^\gamma F)(x) \mathbf{1} = \pi_0 F(x) \quad \forall F \in \mathcal{D}_\Gamma^\gamma(V).$$

In particular, we have $\Pi_x a \in \mathcal{C}^\vartheta(\mathbb{R}^d)$ for all $a \in V$.

Note that the 'in particular' part indeed follows from the general statement since $\Gamma_{\cdot x} a \in \mathcal{D}_\Gamma^\gamma(V)$ for all γ large enough and

$$\Pi_x a - \Pi_y \Gamma_{yx} a = 0 \quad \forall y \in \mathbb{R}^d,$$

thus $\Pi_x a = \mathcal{R}^\gamma(\Gamma_{\cdot x} a) \in \mathcal{C}^\vartheta(\mathbb{R}^d)$ since the reconstruction of a modeled distribution is characterized by equation (5.2).

5.2 Construction of a regularity structure based on a Hopf algebra

5.2.1 General construction

We closely follow Section 4.3 [Hai14] in this subsection. Let $(H, \diamond, \Delta, \mathcal{S})$ be a locally finite dimensional connected graded Hopf algebra over \mathbb{R} whose grading differentiates into a d -dimensional grading

$$H = \bigoplus_{n \in \mathbb{N}_0^d} H_n$$

for some $d \in \mathbb{N}$, i.e.

$$H_n \diamond H_m \subseteq H_{n+m}, \quad \Delta H_n \subseteq \bigoplus_{0 \leq k \leq n} H_k \otimes H_{n-k}, \quad \mathcal{S} H_n \subseteq H_n \quad \forall n, m \in \mathbb{N}_0^d.$$

With locally finite dimensional we mean that H_n is finite dimensional for all $n \in \mathbb{N}_0^d$. The notion of a d -dimensional grading of course reduces to the usual notion of a grading in the case $d = 1$ (see Definition 2.17). Also, we obviously get back a familiar one dimensional grading $(\check{H}_i)_{i \in \mathbb{N}_0}$ by putting

$$\check{H}_i := \bigoplus_{n \in \mathbb{N}_0^d: |n|=i} H_n.$$

Now, choose scaling factors $\alpha = (\alpha_i)_{i=1}^d$, where $\alpha_i \in \mathbb{R}^+$, and put $\langle \alpha, n \rangle := \sum_{i=1}^d \alpha_i n_i$ for all $n \in \mathbb{N}_0^d$. Then, we may introduce the index set A and the model space T as

$$A := \bigcup_{n \in \mathbb{N}_0^d} \{ \langle \alpha, n \rangle \}, \quad T_\gamma := \bigoplus_{n \in \mathbb{N}_0^d: \langle \alpha, n \rangle = \gamma} H_n, \quad T := \bigoplus_{\gamma \in A} T_\gamma.$$

Let G denote the set of characters on $T = H$, i.e.

$$G := \{ g \in H^* \mid g(x \diamond y) = g(x)g(y) \forall x, y \in H \}.$$

G is a group together with the convolution product $*$ on $H^* = L(H, \mathbb{R})$. Then, define linear maps $\bar{\Gamma}^{l/r} : H^* \rightarrow L(H, H)$ by

$$\bar{\Gamma}_g^l := (g \otimes \text{id})\Delta$$

and

$$\bar{\Gamma}_g^r := (\text{id} \otimes g)\Delta.$$

In both cases, we have the following.

Theorem 5.14. (Based on Section 4.3 [Hai14]) $(A, T, \bar{\Gamma}_g^l)$ and $(A, T, \bar{\Gamma}_g^r)$ are regularity structures with $\min A = 0$. $\bar{\Gamma}^l : G \rightarrow \bar{\Gamma}_g^l$ is a group antiisomorphism and $\bar{\Gamma}^r : G \rightarrow \bar{\Gamma}_g^r$ is a group isomorphism. The structure groups have the characterizations

$$\bar{\Gamma}_G^l = \{ \Gamma \in L(H, H) \mid \Gamma m_\bullet = m_\bullet(\Gamma \otimes \Gamma), \Delta \Gamma = (\Gamma \otimes \text{id})\Delta \}$$

resp.

$$\bar{\Gamma}_G^r = \{ \Gamma \in L(H, H) \mid \Gamma m_\bullet = m_\bullet(\Gamma \otimes \Gamma), \Delta \Gamma = (\text{id} \otimes \Gamma)\Delta \}.$$

In particular, \diamond is a product on the whole $T \times T$.

Proof. (Based on Section 4.3 [Hai14]) We only show the case of $\bar{\Gamma}_G^l$, the other case is completely analogous.

1. $\bar{\Gamma}^l$ is a group antiisomorphism. For all $g, h \in G$, we have

$$\begin{aligned} \bar{\Gamma}_g^l \bar{\Gamma}_h^l &= (g \otimes \text{id})\Delta(h \otimes \text{id})\Delta = (h \otimes g \otimes \text{id})(\text{id} \otimes \Delta)\Delta = (h \otimes g \otimes \text{id})(\Delta \otimes \text{id})\Delta \\ &= ((h * g) \otimes \text{id})\Delta = \bar{\Gamma}_{h * g}^l. \end{aligned}$$

2. $(A, T, \bar{\Gamma}_g^l)$ is a regularity structure. The only thing left to show is (5.1). Let $g \in G$, $n \in \mathbb{N}_0^d$ and $x \in H_n$ be arbitrary. Then, $x \in T_{\langle \alpha, n \rangle}$ and by the multi grading of Δ ,

$$\bar{\Gamma}_g^l x - x = (g \otimes \text{id})\Delta x - x = ((g - \varepsilon) \otimes \text{id})\Delta x \in \bigoplus_{m < n} H_m = T_{< \langle \alpha, n \rangle},$$

since $(g - \varepsilon)\mathbf{1} = 0$ because g is in G .

3. The given characterization of $\bar{\Gamma}_G^l$ holds. First of all, for any $g \in G$, we have

$$\begin{aligned} \bar{\Gamma}_g^l m_\bullet &= (g \otimes \text{id})\Delta m_\bullet = (g \otimes \text{id})(m_\bullet \otimes m_\bullet)\tau_{1324}(\Delta \otimes \Delta) = m_\bullet(g \otimes g \otimes \text{id} \otimes \text{id})\tau_{1324}(\Delta \otimes \Delta) \\ &= m_\bullet(g \otimes \text{id} \otimes g \otimes \text{id})(\Delta \otimes \Delta) = m_\bullet(\bar{\Gamma}_g^l \otimes \bar{\Gamma}_g^l). \end{aligned}$$

and

$$\Delta \bar{\Gamma}_g^l = \Delta(g \otimes \text{id})\Delta = (g \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta)\Delta = (g \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})\Delta = (\bar{\Gamma}_g^l \otimes \text{id})\Delta.$$

Thus, we have shown the ' \subseteq ' part of the characterization. To show the ' \supseteq ' part, let $\Gamma \in \mathbb{L}(H, H)$ such that $\Gamma m_* = m_*(\Gamma \otimes \Gamma)$ and $\Delta \Gamma = (\Gamma \otimes \text{id})\Delta$ be arbitrary. Put $h := \varepsilon \Gamma$. Then,

$$hm_* = \varepsilon \Gamma m_* = \varepsilon m_*(\Gamma \otimes \Gamma) = (\varepsilon \otimes \varepsilon)(\Gamma \otimes \Gamma) = (h \otimes h),$$

which implies that $h \in G$. Also,

$$\Gamma = (\varepsilon \otimes \text{id})\Delta \Gamma = (\varepsilon \otimes \text{id})(\Gamma \otimes \text{id})\Delta = (h \otimes \text{id})\Delta = \bar{\Gamma}_h^l.$$

Hence, the ' \supseteq ' part is also shown. □

5.2.2 Polynomial case

This subsection is a more explicit formulation of what is described mostly in words in Remark 4.19 [Hai14]. Considering the polynomial Hopf algebra \mathcal{P}_d , we have an obvious d -dimensional grading given by $(\langle X^n \rangle)_{n \in \mathbb{N}_0^d}$. But, as it's natural to choose all scaling factors $\alpha_i = 1$, we get $A = \mathbb{N}_0$ and $T = \bigoplus_{k \in A} T_k$ with $T_k = \mathcal{P}_{d,k}$. We identify \mathcal{P}_d^* with the space of formal series of differential operators $(\mathbb{D}_d)_\Pi$ via the duality pairing (3.3), writing

$$\langle D | (p) := \langle D, p \rangle$$

for $D \in (\mathbb{D}_d)_\Pi$, $p \in \mathcal{P}_d$.

$$\begin{aligned} \bar{\Gamma}_{\langle \partial^n |} X^m &= (\langle \partial^n | \otimes \text{id}) \Delta_\circ X^m = \sum_{0 \leq l \leq m} \binom{m}{l} \underbrace{(\partial^n X^l)(0)}_{=\delta_{n,l} n!} X^{m-l} = \begin{cases} \frac{m!}{(m-n)!} X^{m-n} & \text{if } n \leq m \\ 0 & \text{if } n \not\leq m \end{cases} \\ &= \partial^n X^m \end{aligned}$$

Thus, for a formal series D of differential operators, $D \in (\mathbb{D}_d)_\Pi \subset \mathbb{L}(\mathcal{P}_d, \mathcal{P}_d)$, we have

$$\bar{\Gamma}_{\langle D |} = D.$$

Since the group of characters G can be written as

$$G = \{ \langle D | \mid D \in \exp_\circ(\mathbb{D}_{d,1}) \},$$

we may write the structure group as

$$\bar{\Gamma}_G = \exp_\circ(\mathbb{D}_{d,1}),$$

i.e. the structure group consists of the composition exponentials of first order differential operators. Since

$$\exp_\circ\left(\sum_{i=1}^d a_i \partial_i\right) = \exp_\circ(a_1 \partial_1) \cdots \exp_\circ(a_d \partial_d)$$

due to commutativity of (\mathbb{D}_d, \circ) ,

$$\exp_\circ(a_i \partial_i) X_j = \sum_{k=0}^{\infty} \frac{a_i^k}{k!} \partial_i^k X_j = X_j + \delta_{i,j} a_i$$

and $\Gamma(x \bullet y) = \Gamma x \bullet \Gamma y$ for all $\Gamma \in \bar{\Gamma}_G$, we have

$$\exp_\circ\left(\sum_{i=1}^d a_i \partial_i\right) X^n = \prod_{i=1}^d (X_i + a_i)^{n_i} =: (X + a)^n \quad \forall a \in \mathbb{R}^d,$$

i.e. the structure group $\bar{\Gamma}_G$ is nothing but the group of translations of the d -dimensional polynomials.

5.2.3 Connes-Kreimer Hopf algebra case

This section is basically a short formulation of what was suggested in Remark 4.25 [Hai14]. Starting from the Hopf algebra $(\langle \mathcal{F}_d \rangle, \odot, \Delta_*, \mathcal{S}_\odot^*)$, the one dimensional grading and the scaling factor γ , we get a regularity structure $\mathcal{T}^f = (A^f, T^f, \bar{\Gamma}_\mathcal{B}^f)$, where \mathcal{B} is the group of characters on $\langle \mathcal{F}_d \rangle$, $A^f = \gamma\mathbb{N}_0$ and of course $T^f = \langle \mathcal{F}_d \rangle$ with $T_{k\gamma}^f = \langle \mathcal{F}_{d,k} \rangle$. The group \mathcal{B} can be identified with $\text{GrLk}(\langle \mathcal{F}_d \rangle, \Delta_\odot)$ via Theorem 3.34. The map $\bar{\Gamma}^f$ is then recursively given by

$$\bar{\Gamma}_g^f \mathbf{1} = \mathbf{1}, \quad \bar{\Gamma}_g^f [\zeta]_i = g([\zeta]_i) \mathbf{1} + [\bar{\Gamma}_g^f \zeta]_i, \quad \bar{\Gamma}_g^f (\zeta_1 \odot \zeta_2) = \bar{\Gamma}_g^f \zeta_1 \odot \bar{\Gamma}_g^f \zeta_2.$$

Let $\mathcal{M}_1^f([0, T])$ denote those models $(\Pi, \Gamma) \in \mathcal{M}_1^f(\mathbb{R})$ such that $\Gamma_{st} = \text{id}$ for $s, t \in (-\infty, 0]$ and for $s, t \in [T, \infty)$.

Theorem 5.15. (Based on Section 4.4 [Hai14]) *There is a bijective map $I : \hat{\mathcal{R}}^\gamma(\mathbb{R}, \mathbb{R}^d) \rightarrow \mathcal{M}_1^f([0, T])$ which maps a branched rough path $\hat{\mathbf{X}}$ to the unique model $(\Pi, \Gamma) \in \mathcal{M}_1^f([0, T])$ such that*

$$(\Pi_s x)(t) := \hat{\mathbf{X}}_{st}(x), \quad \Gamma_{st} x := (\hat{\mathbf{X}}_{ts} \otimes \text{id}) \Delta_* x = \sum_{(x)}^* \hat{\mathbf{X}}_{ts}(x_1) x_2 \quad \forall s, t \in [0, T], x \in \langle \mathcal{F}_d \rangle \quad (5.3)$$

Proof. Let $\hat{\mathbf{X}} \in \hat{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ be arbitrary. By Theorem 4.13, there is a unique $\bar{\Gamma}_G^f$ -bimap Γ over \mathbb{R} such that it fulfills (5.3) and such that $\Gamma_{st} = \text{id}$ for $s, t \in (-\infty, 0]$ and for $s, t \in [T, \infty)$ (just put $\Gamma_{st} = \Gamma_{0t}$ if $s < 0$ and $\Gamma_{st} = \Gamma_{Tt}$ if $s > T$). Furthermore, there is a unique Π on \mathbb{R} such that (5.3) holds and such that $\Pi_s = \Pi_t \Gamma_{ts}$ for $s, t \in \mathbb{R}$, in particular $\Pi_s = \Pi_0$ if $s < 0$ and $\Pi_s = \Pi_T$ if $s > T$. For this Π and for every $k \in \mathbb{N}_0$, there is a $C_k > 0$ such that

$$|(\Pi_s \zeta)(t)| \leq C_k |t - s|^{\gamma|\zeta|} \quad \forall \zeta \in \mathcal{F}_{d,k}, t, s \in \mathbb{R}$$

Thus, putting $f_s := \Pi_s \zeta$ for some $\zeta \in \mathcal{F}_d$, we get

$$\begin{aligned} |(\Pi_s x)(\mathfrak{S}_s^\delta u)| &= \left| \delta^{-1} \int_{\mathbb{R}} f_s(t) u(\delta^{-1}(t - s)) dt \right| = \left| \int_{\mathbb{R}} f_s(s + \delta t) u(t) dt \right| \leq \int_{\mathbb{R}} |f_s(s + \delta t)| |u(t)| dt \\ &\leq C_{|\zeta|} \delta^\alpha T \end{aligned}$$

for all $s \in \mathbb{R}$ and $\delta \in (0, 1]$. By linearity and local finite dimensionality, the analytic condition for Π in Definition 5.8 of a model follows. The map I is therefore well-defined. It is also injective, since $\hat{\mathbf{X}}_{st} = (\Pi_s \cdot)(t)$ for all $s, t \in [0, T]$. It only remains to show surjectivity.

Let $(\Pi, \Gamma) \in \mathcal{M}_1^f([0, T])$ be arbitrary. Then, due to Theorem 5.13, $\Pi_s x$ is a continuous function for all $s \in \mathbb{R}$ and $x \in \langle \mathcal{F}_d \rangle$. Due to continuity and the analytic condition for Π in Definition 5.8, we have $(\Pi_t \zeta)(t) = 0$ for all $\zeta \in \mathcal{F}_d \setminus \{\mathbf{1}\}$ and $t \in \mathbb{R}$. Since Γ is a $\bar{\Gamma}_G^f$ -bimap, for all $k \in \mathbb{N}_0$ there is a $C_k > 0$ such that

$$|(\Pi_s \zeta)(t)| = |(\Pi_t \Gamma_{ts} \zeta)(t)| = \|\Gamma_{ts} \zeta\|_0 \leq C_{|\zeta|} |t - s|^{\gamma|\zeta|}.$$

Finally, due to Theorem 4.13 there is a $\hat{\mathbf{X}} \in \hat{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$ such that (5.3) holds for Π and Γ . \square

Definition 5.16. (Reformulation of Definition 3.2. [HK14]) For a branched rough path $\hat{\mathbf{X}} \in \hat{\mathcal{R}}^\gamma([0, T], \mathbb{R}^d)$, the model $(\Pi, \Gamma) = I(\hat{\mathbf{X}})$ and $\mathbf{n} = \lfloor \gamma^{-1} \rfloor$, we call $\mathcal{D}_\Gamma^{\mathbf{n}\gamma}$ the space of *controlled rough paths* of $\hat{\mathbf{X}}$.

Applying Theorem 5.10 to $\mathcal{D}_\Gamma^{\mathbf{n}\gamma}$ yields exactly the one-dimensional case of the composition formula for controlled rough paths given in Equation (3.8) [HK14].

5.2.4 Shuffle-deconcatenation Hopf algebra case

The same can be done for the Hopf algebra $(\langle \mathcal{W}_d \rangle, \mathbb{1}, \Delta_{\otimes}, \mathcal{S}_{\mathbb{1}}^{\otimes})$, yielding a regularity structure $\mathcal{T}^w = (A^w, T^w, \bar{\Gamma}_{G^w}^w)$, where G^w denotes the group of characters on $\langle \mathcal{W}_d \rangle$ and where again $A^w = \gamma \mathbb{N}_0$ and $T^w = \langle \mathcal{W}_d \rangle$ with $T_{k\gamma}^w = \langle \mathcal{W}_{d,k} \rangle$. This structure was already discussed in detail in Section 4.4 [Hai14].

Here, we have the recursion

$$\bar{\Gamma}_g^w \mathbf{1} = \mathbf{1}, \quad \bar{\Gamma}_g^w(w \dot{\otimes} e_i) = g(w \dot{\otimes} e_i) \mathbf{1} + \bar{\Gamma}_g^w w \dot{\otimes} e_i.$$

Due to Theorem 4.7, there is a straightforward analogue to Theorem 5.15 in this case.

5.3 Rough path structures revisited

5.3.1 Motivation

Another way to construct regularity structures for rough paths is to perform an abstract fixed point iteration in the spirit of Section 15.2 [FH14]. The problem we look at is, for smooth functions f_i and drivers x^i with $x_t^i = 0$ for $t \leq 0$,

$$dy_t = \sum_{i=1}^d f_i(y_t) dx_t^i$$

Putting $\xi_t^i := \dot{x}_t^i$, this may formally be reformulated as

$$\dot{y}_t = \sum_{i=1}^d f_i(y_t) \xi_t^i$$

or, as a formal integral equation,

$$y_t - y_0 = \sum_{i=1}^d \int_0^t f_i(y_t) \xi_t^i dt.$$

Introducing the integration kernel $K := \chi_{[0, \infty)}$, we have

$$y_t - y_0 = \left[K * \left(\sum_{i=1}^d f_i(y) \xi^i \right) \right] (t).$$

This may be translated to an abstract equation

$$Y = y_0 \mathbf{1} + \mathcal{K} \dot{Y} \quad \text{and} \quad \dot{Y} = \sum_{i=1}^d f_i(Y) \Xi_i. \quad (5.4)$$

for modeled distributions Y and \dot{Y} . The linear operator \mathcal{K} acting on certain modeled distributions corresponds to the convolution with the kernel K . It will turn out that \mathcal{K} can be written as

$$(\mathcal{K}F)(t) = \mathcal{I}F(t) + (K * \mathcal{R}F)(t) \mathbf{1}$$

for some linear operator \mathcal{I} acting on a sector of the model space which we want to introduce. In view of Theorem 5.10, we will define $f_i(Y)\Xi_i$ as

$$\sum_{k=0}^{\infty} \frac{f_i^{(k)}(Y^{\mathbf{1}})}{k!} (Y - Y^{\mathbf{1}})^k \Xi_i.$$

and later cut off that infinite series at some level. Let therefore $\mathcal{Y} = \bigcup_{n=0}^{\infty} \mathcal{Y}_n$ be the set of symbols necessary to describe Y for any family of smooth functions $(f_i)_{i=1}^d$ and $\dot{\mathcal{Y}} = \bigcup_{n=0}^{\infty} \dot{\mathcal{Y}}_n$ the set of symbols necessary to describe \dot{Y} . Starting from $\mathcal{Y}_0 = \{\mathbf{1}\}$, we obtain the sets by iteratively applying (5.4) and collecting the symbols we have not yet included, i.e.

$$\begin{aligned} \dot{\mathcal{Y}}_0 &= \{\Xi_i | i \in I\}, & \mathcal{Y}_1 &= \{\mathcal{I}(\Xi_i) | i \in I\}, \\ \dot{\mathcal{Y}}_1 &= \{\mathcal{I}(\Xi_{i_1}) \cdots \mathcal{I}(\Xi_{i_m}) \Xi_j | m \in \mathbb{N}, i_k, j \in I\}, & \mathcal{Y}_2 &= \{\mathcal{I}(\mathcal{I}(\Xi_{i_1}) \cdots \mathcal{I}(\Xi_{i_m}) \Xi_j) | m \in \mathbb{N}, i_k, j \in I\}, \\ &\vdots & &\vdots \\ \dot{\mathcal{Y}}_n &= \{\zeta_1 \cdots \zeta_m \Xi_i | m \in \mathbb{N}, \zeta_j \in \mathcal{Y}_n, i \in I\}, & \mathcal{Y}_{n+1} &= \{\mathcal{I}(\zeta) | \zeta \in \dot{\mathcal{Y}}_n\}, \end{aligned}$$

where $I := \mathbb{N} \cap [1, d]$. Obviously, we may identify

$$\mathcal{Y} \equiv \mathcal{F}_d \cup \{\mathbf{1}\}, \quad \dot{\mathcal{Y}} \equiv \bigcup_{i=1}^d \mathcal{F}_d \Xi_i$$

via

$$\mathcal{I}(\tau_1 \cdots \tau_m \Xi_i) \equiv [\tau_1 \cdots \tau_m]_i$$

5.3.2 Branched rough paths structure

Thus, as a model space, we put

$$T^{\mathbf{b}} = \langle \mathcal{Y} \rangle \oplus \langle \dot{\mathcal{Y}} \rangle = \langle \mathbf{1} \rangle \oplus \langle \mathcal{F}_d \rangle \oplus \bigoplus_{i=1}^d \langle \mathcal{F}_d \Xi_i \rangle$$

with homogenities

$$|\mathbf{1}| = 0, \quad |\Xi_i| = \gamma - 1, \quad |\tau_1 \cdots \tau_m \Xi_i| = |\tau_1| + \cdots + |\tau_m| + \gamma - 1, \quad |[\zeta]_i| = |\mathcal{I}(\zeta \Xi_i)| = |\zeta| + \gamma$$

and hence index set

$$A^{\mathbf{b}} := \gamma \mathbb{N}_0 \cup (\gamma \mathbb{N} - 1).$$

$T^{\mathbf{b}}$ then can be turned into a comodule of $(\langle \mathcal{F}_d \rangle, \Delta_*)$ via

$$\Delta^{\mathbf{b}} : T^{\mathbf{b}} \rightarrow \langle \mathcal{F}_d \rangle \otimes T^{\mathbf{b}}, \quad \Delta^{\mathbf{b}} \zeta \Xi_i = \sum_{(\zeta)}^* \zeta_1 \otimes \zeta_2 \Xi_i, \quad \Delta^{\mathbf{b}} [\zeta]_i = \Delta_* [\zeta]_i \quad (5.5)$$

which allows us to define the action of some $g \in \langle \mathcal{F}_d \rangle^*$ on $T^{\mathbf{b}}$ as

$$\bar{\Gamma}_g^{\mathbf{b}} := (g \otimes \text{id}) \Delta^{\mathbf{b}}. \quad (5.6)$$

Denoting the group of characters on $\langle \mathcal{F}_d \rangle$ as \mathcal{B} , we get a regularity structure $\mathcal{T}^{\mathbf{b}} := (A^{\mathbf{b}}, T^{\mathbf{b}}, \bar{\Gamma}_{\mathcal{B}}^{\mathbf{b}})$ for branched rough paths. It is some kind of a generalisation of the rough paths structure given in Definition 13.5. [FH14] for the case of $\gamma \in (1/3, 1/2]$, though they work with a finite dimensional model space there, which would also be possible here via truncation.

Theorem 5.17. \mathcal{T}^b is a regularity structure. $\langle \mathcal{Y} \rangle$ and $\langle \dot{\mathcal{Y}} \rangle$ are sectors of \mathcal{T}^b .

Proof. The fact that $\bar{\Gamma}_{\emptyset}^b$ is a structure group follows again from the grading of Δ , and the group antimorphism property of $\bar{\Gamma}^b$ (cf. Theorem 5.14). Since Δ^b by definition maps $\langle \mathcal{Y} \rangle$ to $\langle \mathcal{F}_d \rangle \otimes \langle \mathcal{Y} \rangle$ and $\langle \dot{\mathcal{Y}} \rangle$ to $\langle \mathcal{F}_d \rangle \otimes \langle \dot{\mathcal{Y}} \rangle$, they are indeed sectors. \square

There is an obvious group isomorphism ψ given by $\psi(\bar{\Gamma}_g^f) = \bar{\Gamma}_g^b$, which is also a structure group morphism. It can be expanded to an injective map

$$\Phi : \mathcal{M}^f(\mathbb{R}) \rightarrow \mathcal{M}^b(\mathbb{R}), \quad \Phi(\Pi, \Gamma) := (\varphi(\Pi), \psi(\Gamma))$$

via

$$\varphi(\Pi)_x \zeta := \Pi_x \zeta \quad \forall \zeta \in \langle \mathcal{Y} \rangle, \quad \varphi(\Pi)_x \zeta := \frac{d}{dt} \Pi_x \zeta \quad \forall \zeta \in \langle \dot{\mathcal{Y}} \rangle.$$

We have that

$$\Phi(\mathcal{M}_1^f(\mathbb{R})) = \{(\Pi, \Gamma) \in \mathcal{M}_1^b(\mathbb{R}) \mid \frac{d}{dt} \Pi_x \upharpoonright_{\langle \mathcal{Y} \rangle} = \Pi_x \partial\} =: \mathcal{M}_{\text{ad}}^b(\mathbb{R}),$$

where $\partial \in L(\langle \mathcal{Y} \rangle, \langle \dot{\mathcal{Y}} \rangle)$ is generated by

$$\partial \mathbf{1} := 0, \quad \partial[\zeta]_i := \zeta \Xi_i.$$

Note that $\partial \Gamma \upharpoonright_{\langle \mathcal{Y} \rangle} = \Gamma \partial$ for all Γ in $\bar{\Gamma}_G^b$. The models in $\mathcal{M}_{\text{ad}}^b(\mathbb{R})$ are called *admissible models*.

5.3.3 Integration against the kernel and solving RDEs

We introduce $\mathcal{I} \in L(\langle \dot{\mathcal{Y}} \rangle, \langle \mathcal{Y} \rangle)$ as

$$\mathcal{I}(\zeta \Xi_i) = [\zeta]_i.$$

.

Theorem 5.18. $\mathcal{I} : \langle \dot{\mathcal{Y}} \rangle \rightarrow \langle \mathcal{Y} \rangle$ is an abstract integration map of order 1 in the sense of Definition 5.7 [Hai14] since

- (i) $\mathcal{I}\langle \dot{\mathcal{Y}} \rangle_\alpha \subseteq \langle \mathcal{Y} \rangle_{\alpha+1}$,
- (ii) $\mathcal{I}\Gamma a - \Gamma \mathcal{I}a \in \langle \mathbf{1} \rangle$ for every $a \in \langle \dot{\mathcal{Y}} \rangle$ and $\Gamma \in \bar{\Gamma}_{\emptyset}^b$.

Proof. We have

$$|\mathcal{I}(\zeta \Xi_i)| = |[\zeta]_i| = |\zeta| + \gamma = |\zeta \Xi_i| + 1$$

and

$$\mathcal{I}\bar{\Gamma}_g^b(\zeta \Xi_i) - \bar{\Gamma}_g^b \mathcal{I}(\zeta \Xi_i) = \mathcal{I}[(\bar{\Gamma}_g^b \zeta) \Xi_i] - \bar{\Gamma}_g^b [\zeta]_i = [\bar{\Gamma}_g^b \zeta]_i - g([\zeta]_i) \mathbf{1} - [\bar{\Gamma}_g^b \zeta]_i = -g([\zeta]_i) \mathbf{1}.$$

\square

In the following, let $\odot : \langle \mathcal{Y} \rangle \times \langle \dot{\mathcal{Y}} \rangle \rightarrow \langle \dot{\mathcal{Y}} \rangle$ be generated by

$$\tau \odot (\zeta \Xi_i) := (\tau \odot \zeta) \Xi_i \quad \forall \tau \in \mathcal{F}_d \cup \{\mathbf{1}\}, \zeta \in \mathcal{F}_d, i \in \{1, \dots, d\}.$$

Theorem 5.19. A linear map $\Gamma \in L(T^b, T^b)$ is in $\bar{\Gamma}_{\emptyset}^b$ if and only if all of the following properties are satisfied:

- (i) $\Gamma x - x \in T_{<\alpha}^b$ for all $\alpha \in A^b$ and $x \in T_\alpha^b$,

- (ii) $\Gamma\langle\mathcal{Y}\rangle \subseteq \langle\mathcal{Y}\rangle$ and $\Gamma\langle\dot{\mathcal{Y}}\rangle \subseteq \langle\dot{\mathcal{Y}}\rangle$,
 (iii) $\mathcal{I}\Gamma b - \Gamma\mathcal{I}b \in \langle\mathbf{1}\rangle$ for every $b \in \langle\dot{\mathcal{Y}}\rangle$,
 (iv) $\Gamma a \odot \Gamma b = \Gamma(a \odot b)$ for all $a \in \langle\mathcal{Y}\rangle$ and $b \in \langle\dot{\mathcal{Y}}\rangle$.

Proof.

1. Let $\Gamma \in \bar{\Gamma}_{\mathcal{B}}^b$ be arbitrary. Let $g \in \mathcal{B}$ be such that $\Gamma = \bar{\Gamma}_g^b$. Properties (i) and (ii) hold by Theorem 5.17, property (iii) holds by Theorem 5.18 and property (iv) holds since

$$\begin{aligned} \Gamma(\tau \odot \zeta \Xi_i) &= (g \otimes \text{id})\Delta^b(\tau \odot \zeta \Xi_i) = ((g \otimes \text{id})\Delta_*(\tau \odot \zeta))\Xi_i = ((g \otimes \text{id})(\Delta_*\tau \odot \Delta_*\zeta))\Xi_i \\ &= ((g \otimes \text{id})\Delta_*\tau) \odot ((g \otimes \text{id})\Delta_*\zeta)\Xi_i = ((g \otimes \text{id})\Delta^b\tau) \odot ((g \otimes \text{id})\Delta^b(\zeta \Xi_i)) \\ &= \Gamma\tau \odot \Gamma(\zeta \Xi_i) \end{aligned}$$

2. Let $\Gamma \in L(T^b, T^b)$ be such that it satisfies all of the given properties. Denote by g the element in $\langle\mathcal{F}_d\rangle^*$ recursively generated by

$$g(\mathbf{1}) := 1, \quad g([\zeta]_i \mathbf{1}) := \Gamma\mathcal{I}(\zeta \Xi_i) - \mathcal{I}\Gamma(\zeta \Xi_i), \quad g(\zeta_1 \odot \zeta_2) := g(\zeta_1)g(\zeta_2).$$

By the first and last equation, we get that $g \in \mathcal{B}$. The claim we want to show is $\Gamma = \bar{\Gamma}_g^b$. By (5.6) and (5.5) this is equivalent to the identities

$$\Gamma\tau = (g \otimes \text{id})\Delta_*\tau \tag{5.7}$$

for all $\tau \in \mathcal{F}_d \cup \{\mathbf{1}\}$ and

$$\Gamma(\zeta \Xi_i) = ((g \otimes \text{id})\Delta_*\zeta)\Xi_i \tag{5.8}$$

for all $\zeta \in \mathcal{F}_d$.

First of all, we indeed have

$$\Gamma\mathbf{1} = \mathbf{1} = (g \otimes \text{id})\Delta_*\mathbf{1}$$

by properties (i) and (ii) as well as

$$\Gamma\Xi_i = \Xi_i = ((g \otimes \text{id})\Delta_*\mathbf{1})\Xi_i$$

by property (i) and the fact that the Ξ_i have the lowest homogeneity of the regularity structure. Thus, recalling the iterative scheme from the motivation, (5.7) holds on \mathcal{Y}_0 and (5.8) holds on \mathcal{Y}_0 .

Assuming (5.8) holds on $\dot{\mathcal{Y}}_n$, we get

$$\Gamma[\zeta]_i = \Gamma\mathcal{I}(\zeta \Xi_i) = g([\zeta]_i \mathbf{1}) + \mathcal{I}\Gamma(\zeta \Xi_i) = g([\zeta]_i \mathbf{1}) + (g \otimes [\cdot]_i)\Delta_*\zeta = (g \otimes \text{id})\Delta_*[\zeta]_i$$

for all $\zeta \in \dot{\mathcal{Y}}_n$, using in the second equality the definition of g and in the last equality the definition of Δ_* given in (3.22). Thus, it follows that (5.7) holds on \mathcal{Y}_{n+1} .

Assuming that (5.7) holds on \mathcal{Y}_{n+1} , we show that (5.8) holds on $\dot{\mathcal{Y}}_{n+1}$ by an induction over the number of trees of a symbol in $\dot{\mathcal{Y}}_{n+1}$. Indeed, we first have for all $\tau \in \mathcal{Y}_{n+1}$ that

$$\Gamma(\tau \Xi_i) = \Gamma\tau \odot \Gamma\Xi_i = (\Gamma\tau)\Xi_i = ((g \otimes \text{id})\Delta_*\tau)\Xi_i,$$

where we used property (iv). Then, if (5.8) holds for some $\zeta \Xi_i \in \dot{\mathcal{Y}}_{n+1}$, we get for all $\tau \in \mathcal{Y}_{n+1}$ that

$$\begin{aligned} \Gamma(\tau \odot \zeta \Xi_i) &= \Gamma\tau \odot \Gamma(\zeta \Xi_i) = (g \otimes \text{id})\Delta_*\tau \odot ((g \otimes \text{id})\Delta_*\zeta)\Xi_i = ((g \otimes \text{id})(\Delta_*\tau \odot \Delta_*\zeta))\Xi_i \\ &= ((g \otimes \text{id})\Delta_*(\tau \odot \zeta))\Xi_i, \end{aligned}$$

where we used property (iv) again in the first identity, the definition of g in the third identity and in the last identity the definition of Δ_* given in (3.22). Thus, the induction over the number of trees is complete.

Finally, by an induction over n , we obtain the claim. \square

Theorem 5.20. *Let $\alpha > 0$, $f \in C^k(\mathbb{R}, \mathbb{R})$ for some $k \in \mathbb{N}$ with $k \geq \alpha/\gamma + 1$ and $(\Pi, \Gamma) \in \mathcal{M}^b(\mathbb{R})$. Then, $Y \mapsto (f(Y)\Xi_i)_{\alpha+\gamma-1}$, where*

$$(f(Y)\Xi_i)_{\alpha+\gamma-1}(t) := \sum_{n=0}^{\lfloor \alpha/\gamma \rfloor} \pi_{<\alpha+\gamma-1} \frac{f^{(n)}(Y^1(t))}{n!} (Y(t) - Y^1(t)\mathbf{1})^{\odot n} \Xi_i$$

and $Y^1(t)\mathbf{1} := \pi_0 Y(t)$, is a continuous map from $\mathcal{D}_\Gamma^\alpha(\mathbb{R}, \langle \mathcal{Y} \rangle)$ into $\mathcal{D}_\Gamma^{\alpha+\gamma-1}(\mathbb{R}, \langle \mathcal{Y} \rangle)$. More precisely, for any compact $\mathfrak{K} \subseteq \mathbb{R}^d$ and any $D > 0$, there is a $C_{\mathfrak{K}, D} > 0$ such that

$$\|(h(F)\Xi_i)_{\alpha+\gamma-1} - (h(G)\Xi_i)_{\alpha+\gamma-1}\|_{\alpha+\gamma-1, \mathfrak{K}} \leq C_{\mathfrak{K}, D} \|F - G\|_{\alpha, \mathfrak{K}} \quad (5.9)$$

for all $F, G \in \mathcal{D}_\Gamma^\alpha(\mathbb{R}, \langle \mathcal{Y} \rangle)$ such that $\|F\|_{\alpha, \mathfrak{K}} + \|G\|_{\alpha, \mathfrak{K}} \leq D$.

Proof. First extend T^b to $\langle \mathcal{F}_d \rangle \oplus \bigoplus_{i=1}^d \langle \mathcal{F}_d \Xi_i \rangle$ with $\bar{\Gamma}_g^b|_{\langle \mathcal{F}_d \rangle} = \bar{\Gamma}_g^f$ and apply Theorem 5.10 to $\mathcal{D}_\Gamma^\alpha(\mathbb{R}, \langle \mathcal{F}_d \rangle)$ to get $Y \mapsto (f(Y))_\alpha$. For all $G \in \mathcal{D}_\Gamma^\alpha(\mathbb{R}, \langle \mathcal{F}_d \rangle)$ and all compact $\mathfrak{K} \subset \mathbb{R}$, we have

$$\begin{aligned} \sup_{\substack{s, t \in \mathfrak{K}: \\ 0 < |t-s| \leq 1}} \sup_{\beta < \alpha + \gamma - 1} \frac{\|G(t)\Xi_i - \Gamma_{ts}(G(s)\Xi_i)\|_\beta}{|t-s|^{\alpha+\gamma-1-\beta}} &= \sup_{\substack{s, t \in \mathfrak{K}: \\ 0 < |t-s| \leq 1}} \sup_{\beta < \alpha + \gamma - 1} \frac{\|(G(t) - \Gamma_{ts}G(s))\Xi_i\|_\beta}{|t-s|^{\alpha+\gamma-1-\beta}} \\ &= \sup_{\substack{s, t \in \mathfrak{K}: \\ 0 < |t-s| \leq 1}} \sup_{\beta < \alpha + \gamma - 1} \frac{\|G(t) - \Gamma_{ts}G(s)\|_{\beta-\gamma+1}}{|t-s|^{\alpha+\gamma-1-\beta}} \\ &= \sup_{\substack{s, t \in \mathfrak{K}: \\ 0 < |t-s| \leq 1}} \sup_{\theta < \alpha} \frac{\|G(t) - \Gamma_{ts}G(s)\|_\theta}{|t-s|^{\alpha-\theta}} \end{aligned}$$

and

$$\sup_{t \in \mathfrak{K}} \sup_{\beta < \alpha + \gamma - 1} \|G(t)\Xi_i\|_\beta = \sup_{t \in \mathfrak{K}} \sup_{\beta < \alpha + \gamma - 1} \|G(t)\|_{\beta-\gamma+1} = \sup_{t \in \mathfrak{K}} \sup_{\theta < \alpha} \|G(t)\|_\theta,$$

and therefore $\|G\Xi_i\|_{\alpha+\gamma-1, \mathfrak{K}} = \|G\|_{\alpha, \mathfrak{K}}$. Thus, $(f(Y)\Xi_i)_{\alpha+\gamma-1}$ indeed is an element of the space $\mathcal{D}_\Gamma^{\alpha+\gamma-1}(\mathbb{R}, \langle \mathcal{Y} \rangle)$ and we get the Lipschitz bounds (5.9) from those in Theorem 5.10. \square

For the remainder of the discussion, we confine ourselves to a basic outline. The idea is to convolute the kernel K with the reconstructions of modeled distributions in $\mathcal{D}_\Gamma^\alpha(\mathbb{R}, \langle \mathcal{Y} \rangle)$ for some $\alpha > 0$. Since those reconstructions live in $\mathcal{C}^{\gamma-1}(\mathbb{R})$ and K is a 1-regularizing kernel in the sense of Assumption 5.1 [Hai14], it is indeed possible to perform such a convolution and to obtain a continuous function in $\mathcal{C}^\gamma(\mathbb{R})$ as a result, if we further restrict the set of models in order to get sufficient decay at $-\infty$. We do this by simply demanding that $\Gamma_{st} = \text{id}$ for $s, t \leq 0$, thus this restricted set will still contain the usual setting of rough paths where everything happens on a compact interval of the form $[0, T]$. Now, we can define an abstract kernel operator $\mathcal{K}^\alpha : \mathcal{D}_\Gamma^\alpha([0, \infty), \langle \mathcal{Y} \rangle) \rightarrow \mathcal{D}_\Gamma^{\alpha+1}([0, \infty), \langle \mathcal{Y} \rangle)$ via

$$(\mathcal{K}^\alpha F)(t) := \mathcal{I}F(t) + (K * \mathcal{R}^\alpha F)(t)\mathbf{1}.$$

This operator then obviously fulfills the property that

$$\mathcal{R}^{\alpha+1}\mathcal{K}^\alpha F = K * \mathcal{R}^\alpha F \quad \forall F \in \mathcal{D}_\Gamma^\alpha(\langle \mathcal{Y} \rangle).$$

By showing appropriate Lipschitz bounds for $\mathcal{K}^\alpha F$ relative to F , we then see that there is $T > 0$ such that

$$\Psi : \mathcal{D}_\Gamma^{\alpha+1-\gamma}([0, T], \langle \mathcal{Y} \rangle) \rightarrow \mathcal{D}_\Gamma^{\alpha+1-\gamma}([0, T], \langle \mathcal{Y} \rangle), \quad \Psi(Z) := y_s \mathbf{1} + \mathcal{K}^\alpha \sum_{i=1}^d (f_i(Z) \Xi_i)_\alpha$$

is a contraction, and therefore has a unique fixed point.

5.3.4 Geometric rough paths structure

For weakly geometric rough paths, we consider the model space

$$T^\mathfrak{g} = \langle \mathcal{W}_d \rangle \oplus \bigoplus_{i=1}^d \langle \mathcal{W}_d \Xi_i \rangle,$$

since the kernel of the linear map

$$\phi_T : T^\mathfrak{b} \rightarrow T^\mathfrak{g}, \quad \phi_T(\mathbf{1}) := \mathbf{1}, \quad \phi_T(\tau) := \phi(\tau) \quad \forall \tau \in \langle \mathcal{T}_d \rangle, \quad \phi_T(\zeta \Xi_i) := \phi(\zeta) \Xi_i \quad \forall \zeta \in \langle \mathcal{F}_d \rangle,$$

consists of those linear combinations of symbols which will vanish through integration by parts. Here, $\phi : \langle \mathcal{F}_d \rangle \rightarrow \langle \mathcal{W}_d \rangle$ is again the Hopf algebra endomorphism from Theorem 4.14. As for the case of branched rough paths, $T^\mathfrak{g}$ then can be turned into a comodule of now $(\langle \mathcal{W}_d \rangle, \Delta_\otimes)$ via

$$\Delta^\mathfrak{g} : T^\mathfrak{g} \rightarrow \langle \mathcal{W}_d \rangle \otimes T^\mathfrak{g}, \quad \Delta^\mathfrak{g} w \Xi_i = \sum_{(w)}^{\otimes} w_1 \otimes w_2 \Xi_i, \quad \Delta^\mathfrak{g} w = \Delta_\otimes w$$

which allows us to define the action of some $g \in \langle \mathcal{W}_d \rangle^*$ on $T^\mathfrak{g}$ as

$$\bar{\Gamma}_g^\mathfrak{g} := (g \otimes \text{id}) \Delta^\mathfrak{g}.$$

Recalling the group $G^\mathfrak{w}$ of characters on $\langle \mathcal{W}_d \rangle$, we get a regularity structure $(A^\mathfrak{g}, T^\mathfrak{g}, \bar{\Gamma}_{G^\mathfrak{w}}^\mathfrak{g})$ for weakly geometric rough paths. In the case of $\gamma \in (1/3, 1/2]$, it again reduces to the rough paths structure given in Definition 13.5. [FH14] after a suitable truncation. With $\psi(\bar{\Gamma}_g^\mathfrak{g}) := \bar{\Gamma}_{g\phi}^\mathfrak{b}$, (ϕ_T, ψ) can be shown to be a regularity structure morphism and since ϕ_T is surjective and ψ injective, there is a sub regularity structure of $\mathcal{T}^\mathfrak{b}$ we may identify $\mathcal{T}^\mathfrak{g}$ with. Thus, all the concepts developed for $\mathcal{T}^\mathfrak{b}$ in the previous subsections can be translated to the case of $\mathcal{T}^\mathfrak{g}$.

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Frequently used notation

1	Unit element of an algebra, 4
Δ_{\sqcup}	The coproduct which is dual to the shuffle product \sqcup , 20
Δ_*	Connes-Kreimer coproduct of cuts of trees and forests, 37
$\tilde{\Delta}$	The reduced coproduct for some coproduct Δ , 7
$\sum_{(c)} c_1 \otimes c_2$	Sweedler's notation of some coproduct value Δc , 4
ε	Counit of a coalgebra, 4
$\hat{\tau}$	Flip, 4
m_{\circ}	Linear map corresponding to the bilinear product \diamond , 4
\mathcal{P}_d	Space of polynomials in d dimensions, 16
S	Antipode of an Hopf algebra, 7
$T(V)$	Tensor algebra of some vector space V , 20
u	Unit map of an algebra, 4
V^*	Algebraic dual space of the vector space V , 3
\mathcal{W}_I	Set of words over an alphabet I , 44
$\langle \cdot, \cdot \rangle$	A duality pairing of two vector spaces, 12
$\underline{\diamond}$	Canonical algebra product on $A \otimes A$ for some algebra (A, \diamond) , 5
\sqcup	Shuffle product, 23
\star	Product on trees and forests dual to the coproduct Δ_* , 42

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