Hopf algebras and non-associative algebras in the study of iterated-integral signatures and rough paths

Rosa Lili Dora Preiß

Technische Universität Berlin

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Structure of the thesis

A rough path perspective on renormalization
joint with Yvain Bruned, Ilya Chevyrev and Peter Friz

Signatures of paths transformed by polynomial maps
joint with Laura Colmenarejo

Areas of areas generate the shuffle algebra
joint with Joscha Diehl, Terry Lyons and Jeremy Reizenstein

A rough path perspective on renormalization

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We look at equations with rough driver given by a collection of smooth vector fields $f = (f_1, \ldots, f_d)$.
In the case of the Itô-Stratonovich correction for a d-dimensional Brownian motion B with covariation $[B^i, B^j]_t = C^{ij} t$, we have

$$\mathrm{d}Y = f\,(Y)\,\mathrm{d}_{\mathsf{Strat}} B$$

if and only if

$$\mathrm{d}Y = f\,(Y)\,\mathrm{d}_{\mathsf{Itô}} B + \tfrac{1}{2} \sum_{i,j=1}^{d} C^{ij} (f_i \triangleright f_j)(Y) dt.$$

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Recall...

Definition (e.g. Hairer-Kelly '14)

A d-dimensional γ-Hölder *weakly geometric rough path* is a map $\mathbf{X} : [0,T]^2 \to \mathrm{T}((\mathbb{R}^d))$ such that

1. $\langle \mathbf{X}_{st}, \mathbf{1} \rangle = 1$ and $\langle \mathbf{X}_{st}, w \sqcup\!\sqcup v \rangle = \langle \mathbf{X}_{st}, w \rangle \langle \mathbf{X}_{st}, v \rangle$ for all words $w, v \in \mathrm{T}(\mathbb{R}^d)$,
2. $\langle \mathbf{X}_{st}, w \rangle = \langle \mathbf{X}_{su} \otimes \mathbf{X}_{ut}, \Delta_{\bullet} w \rangle$ for all words $w \in \mathrm{T}(\mathbb{R}^d)$,
3. $\sup_{s \neq t} \frac{|\langle \mathbf{X}_{st}, w \rangle|}{|t-s|^{\gamma |w|}} < \infty$ for all words $w \in \mathrm{T}(\mathbb{R}^d)$.

Definition

$$\mathrm{d}Y = f(Y)\mathrm{d}\mathbf{X} \quad \Leftrightarrow \quad Y_t - Y_s = \sum_{|w| \leq \gamma^{-1}} f_w(Y_t) \langle \mathbf{X}_{st}, w \rangle + o(|t-s|)$$

where $f_{\mathtt{i} \bullet w} = f_{\mathtt{i}} \triangleright f_w$, $f_{\mathtt{i}} = f_i$

Recall...

Definition (e.g. Hairer-Kelly '15)

A d-dimensional γ-Hölder *branched rough path* is a map $\mathbf{X} : [0,T]^2 \to \mathcal{H}^*$ such that

1. $\langle \mathbf{X}_{st}, \mathbf{1} \rangle = 1$ and $\langle \mathbf{X}_{st}, \zeta_1 \odot \zeta_2 \rangle = \langle \mathbf{X}_{st}, \zeta_1 \rangle \langle \mathbf{X}_{st}, \zeta_2 \rangle$ for all forests $\zeta_1, \zeta_2 \in \mathcal{H}$,
2. $\langle \mathbf{X}_{st}, \zeta \rangle = \langle \mathbf{X}_{su} \otimes \mathbf{X}_{ut}, \Delta_\star \zeta \rangle$ for all forests $\zeta \in \mathcal{H}$,
3. $\sup_{s \neq t} \frac{|\langle \mathbf{X}_{st}, \zeta \rangle|}{|t-s|^{\gamma|\zeta|}} < \infty$ for all forests $\zeta \in \mathcal{H}$.

Definition

$$\mathrm{d}Y = f(Y)\mathrm{d}\mathbf{X} \quad \Leftrightarrow \quad Y_t - Y_s = \sum_{|\tau| \leq \gamma^{-1}} f_\tau(Y_t) \langle \mathbf{X}_{st}, \tau \rangle + o(|t-s|)$$

where $f_{\tau_1 \curvearrowright \tau_2} = f_{\tau_1} \triangleright f_{\tau_2}$, $f_{\bullet_i} = f_i$, $\tau_1 \curvearrowright \tau_2 = \pi_{\mathcal{B}}(\tau_1 \star \tau_2)$

We construct a homomorphism T_v with $T_v \mathtt{i} = \mathtt{i} + v_i$ ($T_v \bullet_i = \bullet_i + v_i$) such that

Theorem (Bruned-Chevyrev-Friz-P. '17)

Let $v = (v_i)_i \subset \mathcal{B}^N$, $\gamma \in (0,1)$ and $\mathbf{X}$ a d-dimensional weakly geometric (branched) rough path. Then, $T_v\mathbf{X}$ is a γ/N-Hölder d-dimensional weakly geometric (branched) rough path.

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Theorem (Bruned-Chevyrev-Friz-P. '17)

In the same setting, Y is an RDE solution to

$$\mathrm{d}Y = f\left(Y\right)\mathrm{d}\left(T_v\mathbf{X}\right)$$

if and only if Y is an RDE solution to

$$\mathrm{d}Y = f^v(Y)\mathrm{d}\mathbf{X}.$$

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Theorem (Bruned-Chevyrev-Friz-P. '17)

Let $v \equiv v_0 \in \mathcal{B}^N$ without a label 0, $\gamma \in (0,1)$ and $\mathbf{X}$ a $d+1$-dimensional weakly geometric (branched) rough path whose zeroth component is time. Then, $T_v \mathbf{X}$ is a $\gamma \wedge N^{-1}$-Hölder $d+1$-dimensional weakly geometric (branched) rough path whose zeroth component is time.

Theorem (Bruned-Chevyrev-Friz-P. '17)

In the same setting, Y is an RDE solution to

$$\mathrm{d}Y = f(Y)\,\mathrm{d}(T_v \mathbf{X})$$

if and only if Y is an RDE solution to

$$\mathrm{d}Y = f(Y)\mathrm{d}\mathbf{X} + f_v(Y)\mathrm{d}t.$$

Weakly geometric case:

$\mathrm{Prim}(\mathrm{T}(\mathbb{R}^d), \Delta_{⧢}) = \mathfrak{g}(\mathbb{R}^d)$ is the free Lie algebra over $\mathbb{R}^d$

$\Rightarrow T_v : \mathrm{T}((\mathbb{R}^d)) \to \mathrm{T}((\mathbb{R}^d))$ is unique as a (continuous) algebra homomorphism

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Branched case:

$\mathrm{Prim}(\mathcal{H}, \Delta_{\odot}) = \mathcal{B}^d$ is the free pre-Lie algebra over $\mathbb{R}^d$

$\Rightarrow T_v : \mathcal{H}^* \to \mathcal{H}^*$ is unique as a continuous algebra homomorphism whose restriction to $\mathcal{B}^d$ is a pre-Lie algebra homomorphism

We construct a set $\mathscr{M}_{[0,T]}$ of admissable models of a regularity structure with renormalisation group $\mathcal{G}_-$ such that

Theorem (Bruned-Chevyrev-Friz-P. '17)

There is a bijective map which maps a branched rough path $\mathbf{X}$ *whose zeroth component is time to the unique admissable model* $(\Pi, \Gamma) \in \mathscr{M}_{[0,T]}$ *with the property that*

$$(\Pi_s \varphi(\tau))(t) = \langle \mathbf{X}_{st}, \tau \rangle \quad \textit{for all } \tau \in \mathcal{B}, \;\; s, t \in [0, T].$$

Theorem (Bruned-Chevyrev-Friz-P. '17)

There is a one-to-one correspondence between elements $v \in \mathcal{B}^{\lfloor 1/\gamma \rfloor}$ *without a label* 0 *and* $\ell \in \mathcal{G}_-$ *such the following diagram commutes*

$$\begin{array}{ccc} \mathbf{X} & \longleftrightarrow & \Pi \\ \downarrow & & \downarrow \\ T_v \mathbf{X} & \longleftrightarrow & \Pi^{M_\ell} \end{array}$$

Signatures of paths transformed by polynomial maps

joint with Laura Colmenarejo

Theorem (Colmenarejo-P. '18)

For $p : \mathbb{R}^d \to \mathbb{R}^m$ a polynomial map with $p(0) = 0$, we have

$$\langle \sigma(p(X)), w \rangle = \langle \sigma(X), M_p w \rangle$$

Here, $M_p(e) = e$ and

$$M_p(w\mathtt{i}) = \sum_{j=1}^{d} (M_p(w) \sqcup\!\sqcup\, k_p^{ij}) \bullet \mathtt{j}$$

where k_p^{ij} is the shuffle polynomial corresponding to the (i, j)-entry of the Jacobian matrix of p.

For any non-empty words w, v

$$w \sqcup\!\sqcup\, v = w \succ v + v \succ w.$$

$$w \succ (v \succ u) = (w \succ v + v \succ w) \succ u \quad \text{(Zinbiel identity)}$$

Theorem (Schützenberger '58)

Indeed, $(\mathrm{T}^{\geq 1}(\mathbb{R}^d), \succ)$ *is the free Zinbiel algebra over* $\mathbb{R}^d$*.*

I.e., for any Zinbiel algebra $(Z, \succcurlyeq)$ and any linear map $L : \mathbb{R}^d \to Z$, there is a unique homomorphism $\Lambda_L : (\mathrm{T}^{\geq 1}(\mathbb{R}^d), \succ) \to (Z, \succcurlyeq)$ such that

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{\ \iota\ } & (\mathrm{T}^{\geq 1}(\mathbb{R}^d), \succ) \\ & {}_{L}\searrow & \downarrow{\scriptstyle \Lambda_L} \\ & & (Z, \succcurlyeq) \end{array}$$

Theorem (Colmenarejo-P. '18)

Let $L : \mathbb{R}^m \to \mathrm{T}^{\geq 1}(\mathbb{R}^d)$ *be a linear map and* $X : [0, L] \longrightarrow \mathbb{R}^d$ *be a piecewise continuously differentiable path with* $X_0 = 0$*. Then, for all non-empty words* w*,*

$$\left\langle \sigma(X^L), w \right\rangle = \langle \sigma(X), \Lambda_L(w) \rangle \,,$$

where $(X^L)^i_t := \left\langle \sigma\left(X|_{[0,t]} \right), L\mathtt{i} \right\rangle$.

or, equivalently,

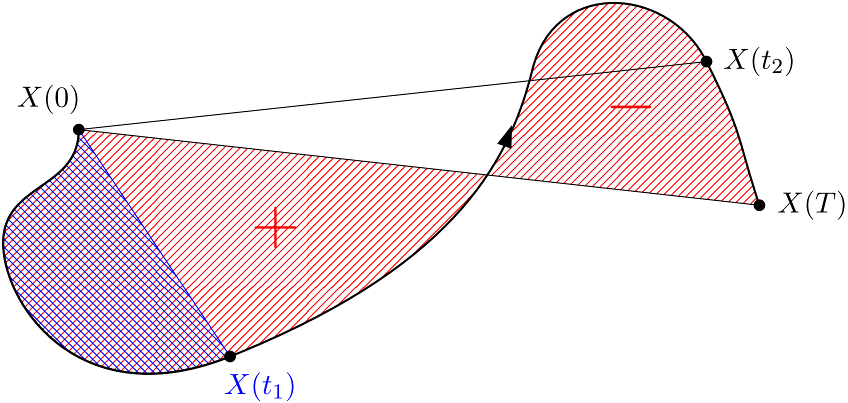
Theorem (e.g. Gehrig-Kawski '08)

$$\int_0^s \mathbf{X}^a_t \mathrm{d}\mathbf{X}^b_t = \mathbf{X}^{a \succ b}_s$$

for any $a, b \in \mathrm{T}^{\geq 1}(\mathbb{R}^d)$*, where* $\mathbf{X}^a_t := \langle \sigma(X|_{[0,t]}), a \rangle$

Areas of areas generate the shuffle algebra

joint with Joscha Diehl, Terry Lyons and Jeremy Reizenstein



$$\mathsf{SignedArea}(X^1, X^2)_t = \frac{1}{2}\left(\int_0^t X_s^1 \mathrm{d}X_s^2 - \int_0^t X_s^2 \mathrm{d}X_s^1\right).$$

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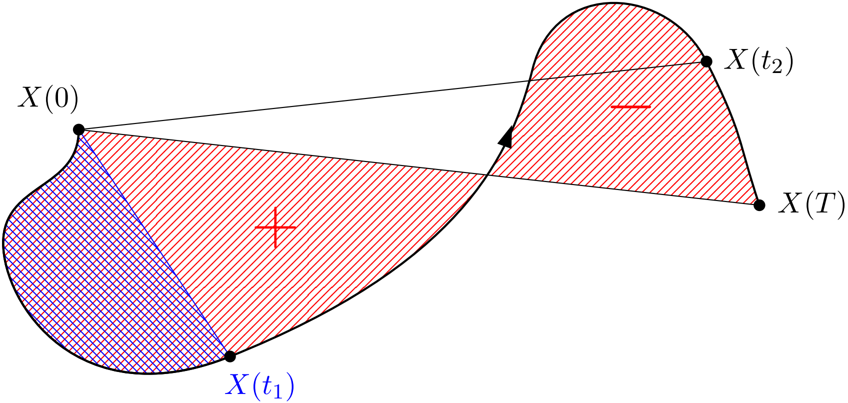

$$\text{Area}(X^1, X^2)_t = \int_0^t X_s^1 \mathrm{d}X_s^2 - \int_0^t X_s^2 \mathrm{d}X_s^1.$$

Define

$$\mathsf{area}(a,b) := a \succ b - b \succ a$$

$$\mathsf{Area}(\mathbf{X}^a, \mathbf{X}^b)_T = \Big\langle \sigma(X), \mathsf{area}(a,b) \Big\rangle.$$

area is non-associative,

$$\mathsf{area}(\mathsf{area}(1,2),3) = 123 - 132 + 213 - 231 - 312 + 321$$
$$\neq 123 - 132 - 213 + 231 - 312 + 321 = \mathsf{area}(1, \mathsf{area}(2,3))$$

area is anticommutative, but doesn't satisfy the Jacobi identity,

$$\mathsf{area}(1, \mathsf{area}(2,3)) + \mathsf{area}(2, \mathsf{area}(3,1)) + \mathsf{area}(3, \mathsf{area}(1,2))$$
$$= -123 + 132 + 213 - 231 - 312 + 321 \neq 0$$

However, area satisfies the so-called *Tortkara identity* introduced by Dzhumadil'daev in 2007:

$$\mathsf{area}(\mathsf{area}(a,b),\mathsf{area}(c,b)) = \mathsf{area}(\mathsf{vol}(a,b,c),b),$$

where

$$\mathsf{vol}(x,y,z) := \mathsf{area}(\mathsf{area}(x,y),z) + \mathsf{area}(\mathsf{area}(y,z),x) + \mathsf{area}(\mathsf{area}(z,x),y).$$

Let $\mathscr{A}(\mathbb{R}^d)$ be the smallest Tortkara subalgebra of $(\mathrm{T}^{\geq 1}(\mathbb{R}^d), \text{area})$ that contains the letters.

Theorem (Dzhumadil'daev-Ismailov-Mashurov '18)

$$\mathscr{A}(\mathbb{R}^d) = \mathbb{R}^d \oplus \bigoplus_{\mathtt{i}<\mathtt{j}} \mathrm{T}(\mathbb{R}^d) \bullet (\mathtt{ij} - \mathtt{ji}).$$

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Applications (Diehl-Lyons-P.-Reizenstein '20, '21):

- $t \mapsto \langle \sigma(X\restriction_{[0,t]}), \phi \rangle$ is piecewise linear for all piecewise linear paths X if and only if $\phi \in \mathbb{R} \oplus \mathscr{A}(\mathbb{R}^d)$
- for a martingale M (linear interpolation M of a discrete martingale) whose Stratonovich expected signature exists for each sub-timeinterval, $t \mapsto \langle \sigma(M\restriction_{[0,t]}), \phi \rangle$ is a martingale (linear interpolation of a discrete martingale)

Theorem (Diehl-Lyons-P.-Reizenstein '20)

$\mathrm{T}^{\geq 1}(\mathbb{R}^d)$ *is shuffle generated by* $\mathscr{A}(\mathbb{R}^d)$*, i.e.*

$$\mathrm{span}\{a_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup a_n : n \geq 1, a_i \in \mathscr{A}(\mathbb{R}^d)\} = \mathrm{T}^{\geq 1}(\mathbb{R}^d)$$

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This is a corollary of the following more general fact.

Theorem (Diehl-Lyons-P.-Reizenstein '20)

Let $X_n \subseteq \mathrm{T}_n(\mathbb{R}^d)$ *and* $X = \bigcup_n X_n$. *Then,*

For all $n \geq 1$, *for all nonzero* $L \in \mathfrak{g}_n$ *there is an* $x \in X_n$ *such that* $\langle x, L \rangle \neq 0$

if and only if

X *shuffle generates the shuffle algebra* $\mathrm{T}(\mathbb{R}^d)$.

Theorem (based on Rocha '03)

$R = \sum_w w \otimes r(w) = \sum_w \rho(w) \otimes w$ *solves* $(\underline{D} - \mathrm{id})R = \frac{1}{2} R \trianglerighteq_{\mathrm{Sym}} R$.

Here, $(p_1 \otimes q_2) \trianglerighteq_{\mathrm{Sym}} (p_2 \otimes q_2) = \mathsf{area}(p_1, p_2) \otimes [q_1, q_2]$,
$\underline{D}(w \otimes v) = |v| w \otimes v$ on words w, v, and r is the Dynkin map.

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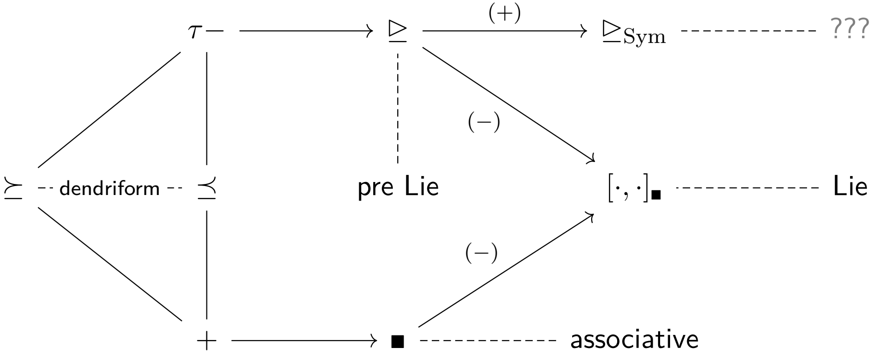

Open Problems

- Do left bracketings of area form a linear basis of $\mathscr{A}(\mathbb{R}^d)$? Is $\mathscr{A}(\mathbb{R}^d)$ free as a Tortkara algebra? Twice yes in $d = 2$!
- How can we build a free generating set of the shuffle algebra in terms of nested monomials of area?
- What is the span of the terms area$(\mathtt{i}_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathtt{i}_n, \mathtt{j}_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathtt{j}_m)$, $n, m \in \mathbb{N}$, $\mathtt{i}_1, \dots \mathtt{i}_n, \mathtt{j}_1, \dots \mathtt{j}_m$ letters? What is their shuffle span together with the letters?
- Can we characterize càdlàg martingales via areas of areas? In the meantime, call random rough paths with vanishing expected signature on $\mathscr{A}(\mathbb{R}^d)$ *martingaloids*.

Thank you.