In how far does the signature characterize the path?

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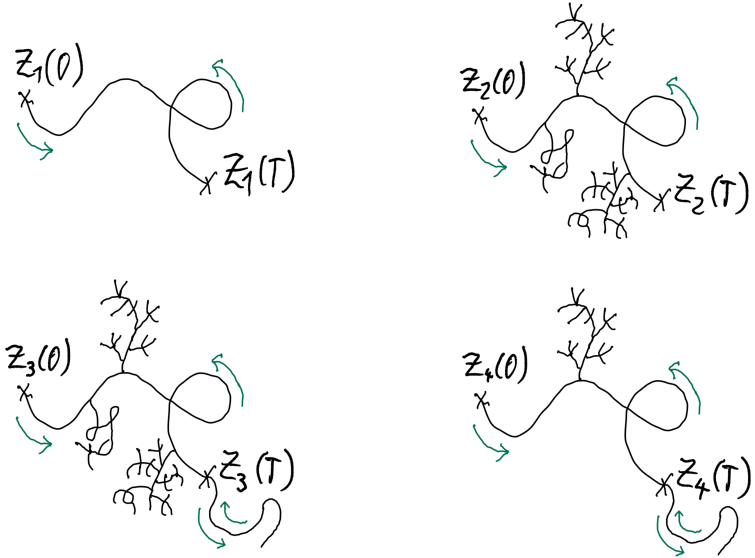
Examples of tree-like equivalent paths



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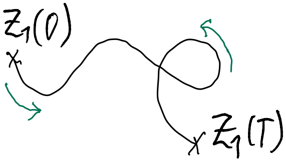
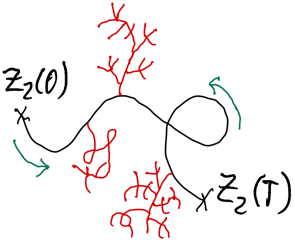
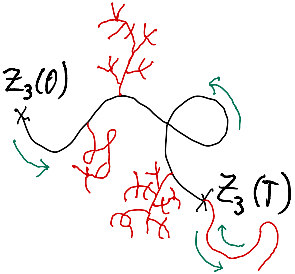
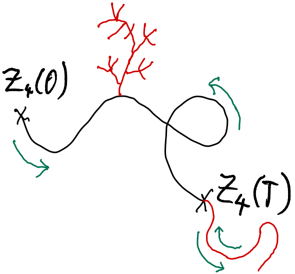
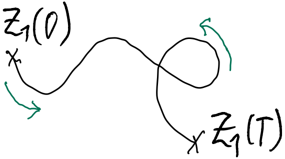


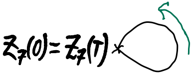


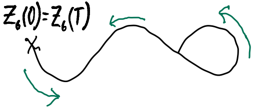


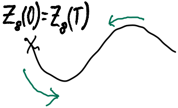


Examples of paths which are not tree-like equivalent









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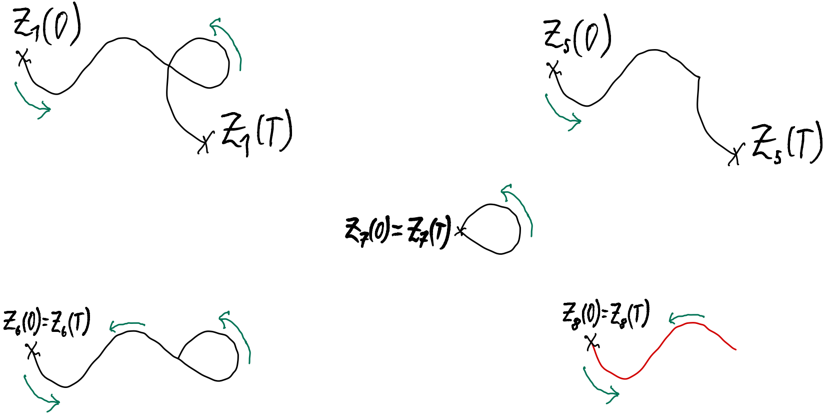
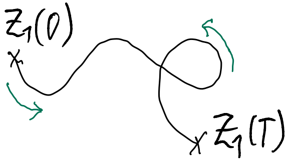
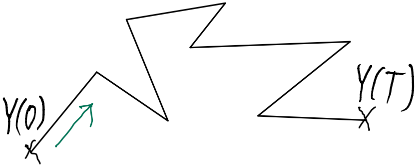


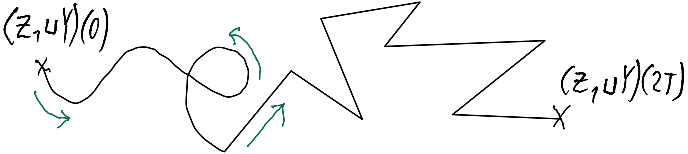
Definition (Hambly-Lyons, Boedihardjo-Geng-Lyons-Yang)

Let (S, O) be a topological space. A continuous map $X : [s, t] \to S$ is called a tree-like path if there is an $\mathbb{R}$-tree τ, a continuous map $\phi : [s, t] \to \tau$ with $\phi(t) = \phi(s)$ and a map $\psi : \tau \to S$ such that $X = \psi \circ \phi$.

Concatenation







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Two continuous maps $X : [s, t] \to S$, $Y : [u, v] \to S$ are called tree-like equivalent if $X \sqcup \overleftarrow{Y} : [s, t + v - u] \to S$ "exists" and is tree-like.

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The tree-like equivalence classes of paths then define the reduced path groupoid/the reduced path group.

A *graded Hopf algebra* is a graded vector space $\mathcal{H}$ together with a unital associative product $\star : \mathcal{H}_n \times \mathcal{H}_m \to \mathcal{H}_{n+m}$ and a counital coassociative coproduct $\Delta : \mathcal{H}_n \to \bigoplus_{m=0}^{n} \mathcal{H}_m \otimes \mathcal{H}_{n-m}$ satisfying certain compatibility relations.

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It is said to be *of finite type* if $\dim(\mathcal{H}_n) < \infty$ for all n.

Let $\mathcal{H}$ be a connected graded cocommutative Hopf algebra of finite type (cgccft Hopf algebra) and

$$P(\mathcal{H}) = \{x \in \mathcal{H} | \Delta x = x \otimes \mathfrak{e} + \mathfrak{e} \otimes x\}$$

Theorem (Milnor-Moore)

$\mathcal{H} \cong \mathcal{U}(P(\mathcal{H}))$, the universal enveloping algebra of the Lie algebra $\mathcal{P}(\mathcal{H})$.

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Corollary

If additionally $\mathcal{P}(\mathcal{H})$ *is a free Lie algebra over a set of homogeneous generators* $(g_i)_{i\in I}$, *then the graded Hopf algebra isomorphism class of* $\mathcal{H}$ *is characterized by the sequence of numbers* $(h_n)_n$,
where h_n *is the number of* g_i *with* $|g_i| = n$.

More precisely, $\mathcal{H} \cong \mathrm{T}^\omega(\mathbb{R}^I)$ *and* $\mathcal{P}(\mathcal{H}) \cong \mathfrak{L}^\omega(\mathbb{R}^I)$, *where* $\omega(\mathtt{i}) = |g_i|$.

Construction of a homogeneous minimal generating set

Let now $\mathcal{H}$ be an arbitrary cgccft Hopf algebra again.

To contruct a minimal generating set, we proceed as follows:

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For level 1, just take any basis of $\mathcal{P}_1(\mathcal{H}) = \mathcal{H}_1$.

For level n, take any (possibly empty) linearly independent set of vectors in $\mathcal{P}_n(\mathcal{H})$ which are furthermore linearly independent from the subset $\sum_{m=1}^{n}[\mathcal{P}_{n-m}(\mathcal{H}), \mathcal{P}_m(\mathcal{H})]$ and linearly generate $\mathcal{P}_n(\mathcal{H})$ together with the brackets from below.

Definition

A p-variation $(\mathcal{H}, \star, \Delta)$-rough path is a map $\mathbf{X} : [0,T]^2 \to \hat{\mathcal{H}}$ such that there exists a control $\nu : [0,T]^2 \to \mathbb{R}$

1. $\mathbf{X}_{tt} = \mathfrak{e}$ and $\mathbf{X}_{st} = \mathbf{X}_{su} \star \mathbf{X}_{ut}$,
2. $\Delta \mathbf{X}_{st} = \mathbf{X}_{st} \otimes \mathbf{X}_{st}$,
3. $\sup_{s \neq t} \frac{\|\mathbf{X}_{st}\|_n^p}{\nu(s,t)^n} < \infty$.

Definition

The signature of $\mathbf{X}$ is the single group-like element $\mathbf{X}_{0T}$.

Theorem (Chen-Chow)

For any $n \in \mathbb{N}$ *and* $x \in \mathfrak{L}_{\leq n}(\mathbb{R}^d)$,
there exists a bounded variation log-signature s *with* $x = \mathrm{proj}_{\leq n} s$.

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Conjecture

For any non-trivial Lie ideal $I \subset \mathfrak{L}(\mathbb{R}^d)$*, there exist two bounded variation log-signatures* $s_1 \neq s_2$ *with* $\operatorname{proj}_{\leq n}(s_1 - s_2) \in I$ *for all* $n > 0$*.*

Theorem

If $\mathcal{H}$ is a connected graded cocommutative Hopf algebra of finite type and $P(\mathcal{H})$ is free over a Lie generating subset,
then $\mathcal{H}$-rough paths are tree-like equivalent if and only if their signature agrees.

Note: This is a generalization of Boedihardjo-Chevyrev, and a consequence of Beodihardjo-Geng-Lyons-Yang.

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Conjecture

If $\mathcal{H}$ is a connected graded cocommutative Hopf algebra of finite type and $P(\mathcal{H})$ is not free over a Lie generating subset,
then there exists a $p \geq 1$ and two non-tree-like-equivalent p-variation rough paths with a common signature.

Proof idea

Take a homogeneous minimal generating set $(g_i)_{i\in I}$ of $\mathcal{P}(\mathcal{H})$ and look at $\mathfrak{L}^{\omega}(\mathbb{R}^I)$.

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If $\mathcal{P}(\mathcal{H})$ is Lie-free, then there is a graded isomorphism $\phi : \mathfrak{L}^\omega(\mathbb{R}^I) \to \mathcal{P}(\mathcal{H})$ and the set of rough paths in both worlds are in bijection. Use Beodihardjo-Geng-Lyons-Yang.

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If $\mathcal{P}(\mathcal{H})$ is not Lie-free, then there is a graded surjective homomorphism $\psi\colon \mathfrak{L}^\omega(\mathbb{R}^I) \to \mathcal{P}(\mathcal{H})$ which is not injective. Construct an example of a rough path in $\mathrm{T}^\omega(\mathbb{R}^I)$ such that under the non-injective map ψ, we see the path, but not the signature.

Definition (Bellingeri-Friz-Paycha-P. 2021)

A $(\mathcal{H}, \star, \Delta)$-smooth rough path of order N is a map $\mathbf{X} : [0,T]^2 \to \hat{\mathcal{H}}$ with

1. $\mathbf{X}_{tt} = \mathfrak{e}$ and $\mathbf{X}_{st} = \mathbf{X}_{su} \star \mathbf{X}_{ut}$,
2. $\Delta \mathbf{X}_{st} = \mathbf{X}_{st} \otimes \mathbf{X}_{st}$,
3. $(s,t) \mapsto \mathbf{X}_{st}$ is smooth,
4. $\dot{\mathbf{X}}_{tt} = \partial_s|_{s=t} \mathbf{X}_{st} \in \mathcal{P}_{\leq N}(\mathcal{H})$

Remark

A smooth rough path of order N is a bounded N-variation rough path.

Definition

A genuinely smooth rough path of order N is a smooth rough path of order N such that there exists a minimal set of homogeneous Lie-generators $(g_i)_i \subset \mathcal{P}_{\leq N}(\mathcal{H})$ with

$$\dot{\mathbf{X}}_{tt} \in \mathrm{span}(g_i)_i$$

for all $t \in [0, T]$.

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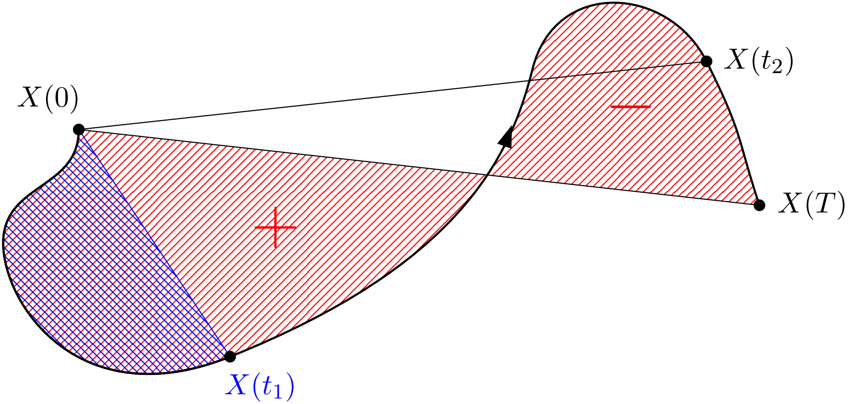
for all $t \in [0, T]$.

Conjecture

If $\mathcal{H}$ is a cgccft Hopf algebra with non-Lie-free primitives, then there is an $N \in \mathbb{N}$ and two genuinely smooth rough paths of order N with the same signature while not being tree-like equivalent.



Picture from Diehl-Lyons-P.-Reizenstein, *Areas of areas generate the shuffle algebra*